Time Domain Finite Element Method for Linear and Nonlinear Models in Electromagnetics and Optics

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M.Sc Asad Anees
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Chairman of the Doctoral Committee:
Prof. Dr. Sven Hartmann, TU Clausthal

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Prof. Dr. Lutz Angermann, TU Clausthal (Advisor)
Prof. Dr. Reza Abedi, UTSI Knoxville, U.S.A.
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Abstract:
In this thesis, we develop a time domain finite element method for linear and nonlinear models in Electromagnetics and Optics. In the linear case, a weak formulation is derived for the electric and magnetic fields with appropriate initial and boundary conditions, and the problem is discretized both in space and time. Nédeléc curl-conforming and Raviart-Thomas div-conforming finite elements are used to discretize in space the electric and magnetic fields, respectively. The backward Euler and symplectic schemes are applied to discretize the linear problem in time. For this linear system, we give a complete stability and error analysis. In addition, computational experiments are presented to validate the method; the electric and magnetic fields are visualized. The method also allows to treat complex geometries of various physical systems coupled to electromagnetic fields in 3D.

In the next part of thesis, we extend the linear finite element method to time domain finite element methods for the full system of Maxwell's equations with cubic nonlinearities in 3D. For the first time, stability and error estimates are presented for this type of problem. The new capabilities of these methods are to efficiently model linear and nonlinear effects of the electrical polarization. The novel strategy has been developed to bring under control the discrete nonlinearity model in space and time. It results in energy stable discretizations both at the semi-discrete and the fully discrete levels, with spatial discretization either using discontinuous spaces and edge elements (Lee-Madsen formulation) or edge and face elements (Nédeléc-Raviart-Thomas formulation). To verify the stability, a novel "nonlinear" electromagnetic energy is introduced, which is stronger than the the commonly used (linear) electromagnetic energy. It turns out that the proposed time discretization scheme is unconditionally stable with respect to this energy. The presented computational experiments demonstrate that the proposed approaches prove to be robust and allow the modeling of 3D optical problems that can be directly derived from the full system of Maxwell's nonlinear equations, and also allow the treatment of complex nonlinearities and geometries of various physical systems.

In the last section of thesis, the time domain discretization for the nonlinear problem is extended to a discontinuous Galerkin finite element method in 2D. The energy of nonlinear Maxwell's equations at the continuous and discrete levels is described, and an error estimate at the semi-discrete level is demonstrated for the discontinuous Galerkin method.

Contents

Introduction

0.1 The Finite Element Method for Maxwell's Equations

The solution of Maxwell's equations in the time domain formulation is involved in many engineering and industrial problems, e.g. RF, radar, mixed signal integrated circuits (ICs), diffraction of electromagnetic waves, plasma physics, acoustic or seismic wave propagation, radiation, scattering, environmental and medical imaging, and microwave devices. In the presence of complex media or geometries, finite element methods in either continuous or discontinuous variants are the main numerical approaches. In the literature an abundance of work about convergence analysis, semi-discrete, fully discrete error estimates, and numerical simulations for the time dependent Maxwell's equations exists. The Galerkin time domain finite element methods (TDFEM) can be grouped into two classes, one class of schemes directly deals with the system of Maxwell's equations, whereas the other class solves the second order wave equations (classical approach).

In the classical approach, second order wave equations or hyperbolic system are obtained either by eliminating the electric or magnetic field from the system of Maxwell's equations. The resulting problems are called electric or magnetic field formulations, respectively. In the literature, a lot of papers have been devoted to the solution of second order wave equations [103, 84, 112, 33, 129, 17, 66, 7, 134, 96, 44, 64, 92, 57, 19]. In [103], Monk presented semi-discrete error estimates in the energy and $\mathbf{L}^2$ norms by employing Nédeléc curl-conforming elements for the vector wave equations. The paper [84] presents TDFEM for the second order vector wave equations and hyperbolic systems using node-based and edge-based elements. Various numerical experiments were performed to investigate the advantages or disadvantages of the mass lumping scheme. Comparisons of various TDFEM and semi-discrete error estimates are presented in [112] for the electric field hyperbolic equation in anisotropic and inhomogeneous media with respect to test and trial spaces, explicit and implicit formulations. Moreover, the convergence of a fully discrete finite element scheme is analyzed for the second order electric field equation (vector wave equation) in [33], and optimal error estimates are obtained in the $\mathbf{L}^2$ norm. Furthermore, the only simulations of the second order vector wave equation are described in [129, 64, 57, 19]. In the papers [33] and [129], the vector wave equation is discretized by Nédeléc curl-conforming finite elements in space, and second order backward and central finite differences, respectively are used to discretize in time. The scheme described by White and Stowell [129] is second order accurate in space and time. Edge finite elements (for the magnetic vector potential) in the time domain are presented in [64] to address the problem of inductive and capacitive effects. A TDFEM (curl-curl electric field equation) forward solver transmitting loop is presented in [92] to address for a complex shaped domain. A local time stepping method (LTS) based on explicit Runge-Kutta schemes having arbitrary accuracy in time for wave propagation is demonstrated in [57].

The vector wave equation and magnetic vector potential approaches allow to address the Maxwell's equations in time domain, an easy implementation for analysis, error estimation and simulations. However, the simulation, analysis and error estimation of the wave equation (electric field formulation and magnetic field formulation) cause spurious and non-physical solutions that are linearly raising corresponding to time [17, 134].

In [102, 105, 100, 73, 86, 120, 88, 89, 71, 85, 139, 99, 127, 138, 78, 135, 9], a number of mixed time domain finite element methods are explained to deal with the system of Maxwell's equations. An abundance of mixed time domain finite element methods for the direct application to Maxwell's system is available, where the electric and magnetic fields are discretized in space by discontinuous and Nédeéc curl-conforming spaces, respectively, see e.g. [102, 86, 88, 89, 138] and [135]. In the work [102], error estimates are demonstrated for a semi-discrete problem, but computational experiments and fully discrete error estimates are not provided. Both semi-discrete, and fully discrete (using a Crank-Nicolson discretization) point-wise super-convergence results are obtained for Maxwell's equations in metamaterials for nonuniform cublic and rectangular edge elements in [69]. Lee and Madsen [86] also demonstrated a mixed time domain finite element simulation for Maxwell's equations and employed a explicit leapfrog time integration scheme. A mixed semi-discrete and a fully discrete error analysis for Maxwell's equations in double negative material are given in [88], but computational experiments were not performed. In [91], a variable time step method for time domain Maxwell's equations is presented. Fully discrete error estimates and computational experiments for Maxwell's equations simultaneously for dispersive and Lorentz metamaterials are presented in [89, 135]. There the temporal discretization is done by means of first order backward finite differences, and computational experiments are performed for 2D situations. Error estimates are also presented in [99], where the problem is discretized in space and time by means of Nédeléc curl-conforming finite elements and backward finite differences, respectively. The scheme is called decoupled or explicit magnetic field scheme and causes spurious solutions. Furthermore, other semi-discrete theoretical and numerical results for the time dependent Maxwell's equation in composite material are presented in [138], where the material parameters ε, μ and σ are 3×3 positive definite matrices depending on the spatial variables.

In [100, 139, 120, 10, 9], time domain finite element methods for Maxwell's equations are discussed using curl-conforming and div-conforming elements for spatial discretization. The $\mathbf{L}^2$ error is estimated for a semi-discrete scheme in [139]. We proposed a splitting approach for the Maxwell's equation that splits the system of Maxwell's equations into two uncoupled system, to deal with ε and μ as matrix function of space (complex material) [8]. The splitting method allows to solve the uncoupled systems independently, and is proved to be stable and convergent at the semi-discrete level. The operator form of uncoupled systems and semi-discrete error estimates are presented in [9], but fully discrete error estimation and computational experiment are not given in [8, 9, 139]. In [100], semi-discrete and fully discrete error estimates are obtained from the operator form of the system of Maxwell's equations, and the time discretizations by rational approximations of the exponential function are investigated, but no computational experiments were given. Moreover, the only simulations for the system of Maxwell's equations are given in [120, 10]

In the article [73], the Maxwell's equations are discretized in space by a node-based and edge-based finite element method, and an efficient solver is described both for the frequency and time domain formulations. The paper [71] describes a general way to investigate the stability of temporal discretization schemes such as backward difference, forward difference and central difference methods in electromagnetics. The stability is determined by analyzing the root locus map of a characteristic equation and evaluating the spectral radius of finite element system matrix. Stability properties are given in [112] for simulations of transient electromagnetic phenomena for the Maxwell's equations. In the article [85], time domain finite element methods based on Whitney elements are proposed for solving transient response problems on tetrahedral meshes. One of the proposed schemes is unconditionally stable, another scheme is explicit but does not require matrix inversions. An explicit time domain finite element algorithm is presented for the Maxwell's equations in [78], for complex media in [127] (only numerical result). An energy conserving method for 3D Maxwell's equations is obtained in [28], based on an exponential operator splitting approach.

Many papers have been written about time domain discontinuous Galerkin (TDDG) methods in computational electromagnetics [2, 41, 31, 45, 47, 58, 93, 123, 124, 121, 21, 65]. Other time domain methods to solve either the Maxwell's equations or the vector wave equation can be found in [43, 107, 77, 104, 63, 82, 49, 14, 18, 115, 122, 27, 140]. Several books for electromagnetics [72, 90, 106, 40, 67, 13] are available for analysis and simulation.

The TDFEM proposed in [99, 100, 8, 9, 91] also cause spurious and non-physical solutions because these methods do not figure out quantities from the system of Maxwell's equations directly. It is well known that $\mathbf{H}^1$ conforming finite elements for electromagnetics may result in spurious and non-physical solutions. The degrees of freedom for Nédeléc curl-conforming and Raviart-Thomas div-conforming finite elements are related to the edges and faces of the meshes, respectively, and not to the mesh nodes. These finite elements also avoid the appearance of non-physical, spurious and divergent solutions [23, 24]. These are good reasons to use Nédeléc curl-conforming and Raviart-Thomas div-conforming finite elements. To date most of contributions have been based on Nédeléc curl-conforming (edge elements) for the time domain solution of Maxwell's equation and a few are using (Nédeléc and Raviart-Thomas) edge and face elements with spurious solutions. The technique of error estimation and simulation we present is motivated by the last three decades works. We demonstrate error estimates for fully discretized Maxwell's equations based on a time domain finite element approach, and simulations using various solvers and visualizations of the computed electric and magnetic fields. In our approach, we deal with the system of Maxwell's equations rather than the second order vector wave equation. Additionally, the electric and magnetic fields figure out directly both in the error estimates and numerical experiments. For simplicity of presentation, the material parameters ε and μ are considered as time independent, piecewise constant scalar functions, but the results can be generalized to more complicated material parameters, e.g. positively definite tensors. The electric and magnetic fields are discretized by Nédeléc curl-conforming and Raviart-Thomas div-conforming finite elements in space, respectively. The properties of these finite elements have been investigated in many articles [110], [111] and [118]. In addition, the problem is discretized in time by backward Euler and sympletic methods. The error analysis of the mixed finite element method for the fully discrete problem is given in the case of only the backward Euler method, and computational results are given in both cases.

Our proposed schemes deal with the system of Maxwell's equations directly in 3D, which cause no spurious solution. Fully discrete error estimates and simulation results with visualizations of the electromagnetic quantities are given. Similar results for fully discrete error estimation could be obtained also for our previous results about the decoupled Maxwell's equations [9], [8]. These are intermediate results that provide a starting point for the development and theoretical-numerical investigation of TDFEM for nonlinear problems in optics and photonics. These include energy-conserving methods in 3D.

0.2 The Finite Element Method for nonlinear Maxwell's Equations in optics and Photonics

Nonlinear optics deals with phenomena that occur when the optical properties of a material change under the action of light. It is a key technology for optical communication, data processing and storage. Nonlinear optical phenomena are nonlinear in the sense that the response of the medium to the light is nonlinearly dependent on its strength. Frequently, the behavior of light waves in a material is modeled by means of a third-order polarization response, that is the polarization $\mathbf{P}(\mathbf{E})$ is a cubic polynomial of the electric field strength $\mathbf{E}$. This modeling approach is widely accepted for not too small, but still moderate intensities. At very high intensities, which shall not be considered here, it is no longer adequate. The books [26, 20, 113, 6] describe the fundamental concepts of nonlinear optics.

Since the investigation of light propagation in nonlinear materials involves the solution of nonlinear partial differential equations, various numerical methods for approximating the solutions dominate the practice, for instance finite difference time-domain (FDTD) methods, slowly varying envelope approximations (SVEA), beam propagation methods (BPM), time-domain finite element methods (TDFEM) – among them time-domain discontinuous Galerkin (TDdG) methods –, pseudo-spectral methods, finite volume methods (FVM) (sometimes also called finite integration techniques (FIT)), and many more.

The more conventional FDTD methods are regarded as robust simulation schemes for linear and nonlinear models in optics and photonics [136, 75, 143, 144, 76, 38, 126, 50, 56, 117, 37, 108], but also generally in the field of Computational Electromagnetism, although there are considerable limitations in terms of applicability to complicated geometries, less smooth data (e.g. caused by material interfaces), etc. Typically the spatial domain is discretized by regular, structured (quadrilateral or hexahedral), staggered grids. The difference scheme presented in [136] served as the basis for one of the most commonly-used methods to solve the linear Maxwell's equations. This scheme is of second order in time and exhibits a significant numerical dispersion over long time interval of wave propagation simulation [38, 37]. FDTD simulations for the full system of nonlinear Maxwell's equations have been presented in [76, 143]. Among other things, interacting waves of different frequencies could be treated directly [76]. The auxiliary differential equation (ADE) method along with finite difference time-domain (FDTD) schemes has been originally employed for linear dispersive materials [75] and for the coupling between the polarization vector and the electric field intensity [55, 143]. This scheme was applied to second- and third-order nonlinear phenomena including spatial soliton propagation [55, 74], linear and nonlinear interface scattering [142], and pulse propagation through nonlinear wave guides [144]. The numerical dispersion over a long time interval for modeling of wave propagation could be overcome by higher-order FDTD methods but in case of curved geometries approximation errors of the boundary ("staircase errors") interfere. Moreover, the FDTD method is only conditionally stable. A lot of interesting modeling and simulation results for linear and nonlinear Lorentz dispersion with nonlinear Kerr response in case of 1D, 2D and 3D can be found in [54, 62, 25, 125, 74, 116, 109]. Among non-standard difference methods, pseudospectral spatial domain schemes have been employed for optical carrier shock [79] and linear Lorentz dispersion with nonlinear response [128] simulation.

Slowly varying envelope approximations are mostly used to simulate effects in nonlinear photonics. Using this scheme, the system of Maxwell nonlinear equations transforms into the nonlinear Schrödinger equation. Various nonlinear effects such as self phase modulation and the Kerr effect can be successfully numerically treated [26, 48]. The beam propagation method with second-order indices of refraction is employed for modeling of nonlinear optical devices exhibiting on-axis behaviour [51].

Finite volume methods have been applied to nonlinear Kerr media in 1D and 2D cases [42, 15]. For the Maxwell's equations with Kerr-type nonlinearity, a hyperbolic system is derived and approximated by the Godunov method. Moreover a higher-order Roe solver is also employed for simulations.

In the past few years, discontinuous Galerkin methods have attracted considerable attention and are now being applied to a wide range of problems from hydrodynamics to acoustics and electrodynamics. To the authors' knowledge, the first mathematical proof for the convergence of the discontinuous Galerkin method when applied to Maxwell's equations was given in the paper [60]. The methods allow a comparatively easy handling of elements of various types and shapes, irregular non-conforming meshes and even locally varying polynomial degree. The continuity of the numerical fields is weakly enforced across mesh interfaces by adding suitable bilinear forms [80]. For the linear situation the papers [30, 35, 2, 31, 60] can be mentioned as examples, for dispersive media we refer to [52, 70, 83, 97].

For the system of Maxwell's equations with material nonlinearities, there are still very few rigorous analysis, error estimates, and simulation results using time domain finite element methods ((TDFEM/TDdG) available [48, 133, 132, 131, 141, 95, 134, 22, 68, 3, 4]. In the paper [22] a higher-order discontinuous Galerkin method is used to discretize the problem in space. Two time discretization schemes are investigated – a second-order leap-frog and the implicit trapezoidal scheme. In the fully discretized problems, the nonlinearity is treated by employing the auxiliary differential equation (ADE) approach. In [22] it has been proved that the first scheme is conditionally stable, while the fully implicit method is unconditionally stable. The results for the proposed schemes were given only for the 1D case, and error estimates were obtained only for the semi-discrete problem under some additional assumptions on the strength of the nonlinearity.

Many of the numerical methods we know for nonlinear Maxwell equations are limited in their applicability due to simplifying assumptions. Therefore, the development of efficient TDFEMs for nonlinear optics and photonics in 3D is a subject of intense research efforts in optics, Engineering and Applied Mathematics.

In this thesis, we present a novel technique to addresss some of the shortcomings. We extend the semi-discrete mixed finite element method [8, 9] and the fully discrete finite element method [10, 12] to the fully time-dependent Maxwell's equations with nonlinearities. The electric and magnetic fields in the Maxwell's equations with a cubic nonlinearity are discretized in space by means of either pairs of discontinuous spaces and Nédeléc curl-conforming spaces or pairs of Nédeléc curl-conforming and Raviart-Thomas div-conforming finite elements. The spatial discretization has all the well-known properties of these spaces [110, 111, 118], especially a high accuracy and the ability to handle complex geometries. In addition to error estimates, we are able to demonstrate that the semi-discrete solutions have similar energy-conserving properties as the exact solution. We also developed a fully discrete scheme for the nonlinear problem in 3D that posses the property of energy stability and is unconditionally stable. Achieving these results required a careful modeling of the nonlinearities in the fully discrete scheme by a suitable application of the auxiliary differential equation (ADE) approach. We could demonstrate fully discrete error estimates as well as fully discrete energy stability. The energy stability proporties are important in the sense that they reflect the physical behaviour of the exact solution and make the schemes robust.

0.3 Thesis Organization

The thesis is structured as follows. The Chapter 1 gives an overview over the notation, spatial discretization and projection operators. Section 1.1 describes basic function spaces and notation. The spatial discretization is discussed in Section 1.2. The projection operators for the Lee-Madsen and the Nédeléc and Raviart-Thomas formulations are discussed in Sections 1.2.1 and 1.2.2, respectively.

In Chapter 2, we deal with the system of Maxwell's equations. Section 2.1 describes the weak formulation and an error estimate for the backward Euler semi-discrete method (Rothe method). In Section 2.2 we describe and investigate the full discretization. Section 2.3 presents a collection of numerical examples. Finally Section 2.4 describes the summary of the Chapter 2.

In Chapter 3, we deal with the system of nonlinear Maxwell's equations in optics and photonics. Sections 3.1 and 3.2 describe the weak formulation and spatial discretization for nonlinear problem, respectively. The energy of nonlinear Maxwell's equations at continuous and discrete level describes in Sections 3.1.1 and 3.2.1, respectively. An error estimate at semi-discrete level demonstrates in the Section 3.3. The fully discretization for the Lee-Madsen and the Nédeléc and Raviart-Thomas formulations is discussed in Section 3.4. In Section 3.5 we describe the error estimate at the fully discrete level. Section 3.6 presents a collection of numerical examples. Finally Section 3.7 describes the summary of the Chapter 3.

In Chapter 4, we demonstrate the TDdG method for the system of nonlinear Maxwell's equations in 2D. Section 4.1 describes the spatial discretization for nonlinear problem. The energy of nonlinear Maxwell's equations at continuous and discrete level is describe in Sections 4.0.1 and 4.1.1, respectively. An error estimate at semi-discrete level is demonstrated in Section 4.2. The full discretization for the discontinuous Galerkin finite element method is discussed in Section 4.3. Finally Section 4.4 describes the summary of the Chapter 3.

Chapter 1

Spaces, Spatial Discretization and Projection Operators

1.1 Spaces and Notation

For a real number $p \geq 1$, the space $L^p(\Omega)$ consists of equivalence classes of Lebesgue-measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |u|^p \, d\mathbf{x} < \infty.$$

If $u \in L^p(\Omega)$, we define its $L^p(\Omega)$-norm as follows :

$$\|u\|_{L^p(\Omega)} = \left\{ \int_\Omega |u(\mathbf{x})|^p \, d\mathbf{x} \right\}^{1/p}.$$

Furthermore, the space $L^\infty(\Omega)$ consists of the equivalence classes of essentially bounded measurable functions $u : \Omega \to \mathbb{R}$ equipped with the norm

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} |u(\mathbf{x})|.$$

The analogous spaces of vector fields $\mathbf{u} : \Omega \to \mathbb{R}^3$ are denoted by $\mathbf{L}^p(\Omega) := [L^p(\Omega)]^3$.

In what follows we have to deal with weighted function spaces. Given a weight $\omega : \Omega \to \mathbb{R}$, where the values of ω are positive a.e. on Ω, we define a weighted inner product and a weighted norm by

$$(\mathbf{u}, \mathbf{v})_\omega := \int_\Omega \omega \, \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} \quad \text{and,}$$
$$\|\mathbf{u}\|_\omega := \|\mathbf{u}\|_{\mathbf{L}^2_\omega(\Omega)} := \sqrt{(\mathbf{u}, \mathbf{u})_\omega},$$

and the space $\mathbf{L}^2_\omega(\Omega)$ consists of vector fields $\mathbf{u} : \Omega \to \mathbb{R}^3$ with Lebesgue-measurable components and such that

$$\|\mathbf{u}\|_\omega < \infty.$$

In the case $\omega = 1$, the subscript is omitted.

As transient problems are addressed, we will work with functions that depend on time and have values in certain Banach spaces. If $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is a vector field of the space variable $\mathbf{x}$ and the time variable t, it is suitable to separate these variables in such a way that $\mathbf{u}(t) = \mathbf{u}(\cdot, t)$ is considered as a function of t with values in a Banach space, say X, with the norm $\|\cdot\|_X$. That is, for any $t \in (0,T)$, the mapping $\mathbf{x} \mapsto \mathbf{u}(\mathbf{x}, t)$ is interpreted as a parameter-dependent element $\mathbf{u}(t)$ of X. In this sense we will write $\mathbf{E}(t) = \mathbf{E}(\cdot, t)$, $\mathbf{H}(t) = \mathbf{H}(\cdot, t)$ and so on.

The space $\mathbf{C}^m(0, T, X)$, $m \in \mathbb{N} \cup \{0\}$, consists of all continuous functions $\mathbf{u} : (0,T) \to X$ that have continuous derivatives up to order m on $(0,T)$. It is equipped with the norm

$$\sum_{j=0}^{m} \sup_{t \in (0,T)} \|\mathbf{u}^{(j)}(t)\|_X.$$

For the sake of consistency in the notation we will write $\mathbf{C}(0,T,X) := \mathbf{C}^0(0,T,X)$.

The space $\mathbf{L}^p(0,T,X)$ with $1 \le p < \infty$ contains (equivalent classes of) strongly measurable functions $\mathbf{u} : (0,T) \to X$ such that

$$\int_0^T \|\mathbf{u}(t)\|_X^p \, dt < \infty$$

(for the definition of strongly measurable functions we refer to [81]). The norm on $\mathbf{L}^p(0,T,X)$ is defined by

$$\|\mathbf{u}\|_{\mathbf{L}^p(0,T,X)} := \left\{ \int_0^T \|\mathbf{u}(t)\|_X^p \, dt \right\}^{1/p}.$$

These spaces can be equipped with a weight, too. In particular, we will write

$$\|\mathbf{u}\|_{\mathbf{L}^2(0,T,\mathbf{L}^2_\omega(\Omega))} := \left\{ \int_0^T \int_\Omega |\mathbf{u}(t)|^2 \, \omega \, d\mathbf{x} dt \right\}^{1/2}.$$

Next we introduce the Sobolev spaces of functions with weak spatial derivatives of maximal order $r \in \mathbb{N}$ in $L^p(\Omega)$, where α is a multi-index:

$$W^{r,p}(\Omega) := \{u \in L^p(\Omega) : \ \partial^\alpha u \in L^p(\Omega) \ \forall \, |\alpha| \le r\}.$$

For $p \geq 1$, the norms and semi-norms are defined by

$$\|u\|_{W^{r,p}(\Omega)}^p := \sum_{|\alpha| \leq r} \int_\Omega \|\partial^\alpha u\|^p \, d\mathbf{x},$$
$$|u|_{W^{r,p}(\Omega)}^p := \sum_{|\alpha| = r} \int_\Omega \|\partial^\alpha u\|^p \, d\mathbf{x}.$$

The modifications for $p = \infty$ are obvious. If $p = 2$, we write $H^r(\Omega) := W^{r,2}(\Omega)$ and $\|\cdot\|_{H^r(\Omega)} := \|\cdot\|_{W^{r,2}(\Omega)}$.

The space $H_0^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^1(\Omega)}$, where $C_0^\infty(\Omega)$ denotes the space of all arbitrarily often differentiable functions with compact support on Ω. It is well knwon that $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ and consists of elements u such that $u = 0$ on Γ in the sense of traces [5]. As in the case of the L^p-spaces, we shall write $\mathbf{W}^{r,p}(\Omega) := [W^{r,p}(\Omega)]^3$ and so on.

Furthermore, we need the following Hilbert spaces that are related to the (weak) rotation and divergence operators:

$$\mathbf{H}(\mathrm{curl}\,;\Omega) := \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \ \nabla \times \mathbf{u} \in \mathbf{L}^2(\Omega)\},$$
$$\mathbf{H}_0(\mathrm{curl}\,;\Omega) := \{\mathbf{u} \in \mathbf{H}(\mathrm{curl}\,;\Omega) : \ \mathbf{u} \times \mathbf{n}|_\Gamma = 0\},$$
$$\mathbf{H}(\mathrm{div};\Omega) := \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \ \nabla \cdot \mathbf{u} \in \mathbf{L}^2(\Omega)\},$$
$$\mathbf{H}_0(\mathrm{div};\Omega) := \{\mathbf{u} \in \mathbf{H}(\mathrm{div};\Omega) : \ \mathbf{u} \cdot \mathbf{n}|_\Gamma = 0\}.$$

These Hilbert spaces are equipped with the norms (resp. induced norms)

$$\|\mathbf{u}\|_{\mathbf{H}(\mathrm{curl}\,;\Omega)} := \left\{\|\mathbf{u}\|^2 + \|\nabla \times \mathbf{u}\|^2\right\}^{1/2},$$
$$\|\mathbf{u}\|_{\mathbf{H}(\mathrm{div};\Omega)} := \left\{\|\mathbf{u}\|^2 + \|\nabla \cdot \mathbf{u}\|^2\right\}^{1/2}.$$

We refer to [118, 94, 53] and [32] for details about these spaces.

1.2 Spatial Discretization

In this section we explain families of finite dimensional subspaces $\mathbf{W}_h \subset \mathbf{W} = \mathbf{L}^2(\Omega)$, $\mathbf{U}_h \subset \mathbf{H}(\mathrm{curl}\,;\Omega)$ and $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div};\Omega)$ that are used to discretize the problems (2.8)–(2.9) and (3.6)–(3.7) in space.

Let $\mathcal{T}_h$ be an arbitrary member of a family of triangulations of Ω consisting of geometric elements K. Each element $K \in \mathcal{T}_h$ is assumed to be an open tetrahedron if K has no face or edge on Γ. The elements that have an edge or face on Γ are allowed to have one curved edge or one curved face, respectively. These elements are called boundary elements [102]. If K is a boundary element K we assign a standard tetrahedron $\tilde{K}$ by connecting the four vertices of K by straight edges.

In addition, all triangulations should be compatible with the discontinuities of the coefficients ε and μ, that is their discontinuities lie on the boundaries of the elements of the triangulations only. Moreover we assume that the family of triangulations is quasi-uniform . That is, there exist constants $\underline{c}_{\mathcal{F}} > 0$, $\bar{c}_{\mathcal{F}} > 0$ independent of K and $\mathcal{T}_h$ such that

$$\underline{c}_{\mathcal{F}} h \leq h_K \leq \bar{c}_{\mathcal{F}} \rho_K \quad \forall K \in \mathcal{T}_h \quad \forall \mathcal{T}_h,$$

where ρ_K is the maximum diameter of the largest ball contained in K or $\tilde{K}$, h_K is the diameter of K and $h := \max_{K \in \mathcal{T}_h} h_K$ [53].

Now let Ω_h be the interior of the set

$$\bigcup_{K \in \mathcal{T}_h} \overline{\tilde{K}}.$$

Let $\mathcal{P}_k$ be the space of scalar real-valued polynomials in three variables of maximal degree k, and $\tilde{\mathcal{P}}_k$ be the space of scalar real-valued homogeneous polynomials of exact degree k. For any $k \in \mathbb{N}$, we define the following subspaces of $\mathbf{P}_k := [\mathcal{P}_k]^3$ (for details see [110], [111] or [106]):

$$\begin{aligned} \mathbf{S}_k &:= \{\mathbf{p} \in [\tilde{\mathcal{P}}_k]^3 :\ \mathbf{x} \cdot \mathbf{p}(\mathbf{x}) = 0\}, \\ \mathbf{R}_k &:= \mathbf{P}_{k-1} \oplus \mathbf{S}_k, \\ \mathbf{D}_k &:= \mathbf{P}_{k-1} \oplus \mathbf{x}\,\tilde{\mathcal{P}}_{k-1}. \end{aligned}$$

Obviously, $\mathbf{S}_k \subset \mathbf{P}_k$ and $\mathbf{R}_k \subset \mathbf{P}_k$.
The space $\mathbf{W}_h$ consists of piecewise polynomial of degree $k-1$,

$$\mathbf{W}_h := \{\mathbf{w} \in \mathbf{L}^2(\Omega) :\ \mathbf{w}|_K \in \mathbf{P}_{k-1}, \forall K \in \mathcal{T}_h\}.$$

In general, the element of $\mathbf{W}_h$ are discontinuous.
Here we describe the so-called first family of Nédélec edge elements. We

mention that in the two definitions it is assumed that a standard reference tetrahedron $\hat{K}$ is used and that an affine transformation between $\hat{K}$ and K is applied.

Definition 1.1 ($\mathbf{R}_k$-unisolvent and curl-conforming dofs) Let K be a tetrahedron in $\mathbb{R}^3$ with faces denoted by f and edges denoted by e. $\mathbf{t}$ and $\mathbf{n}$ in Fig (1.1) represent the unit vectors along the edge e and perpendicular to the face f, respectively. Let $\mathbf{u} \in \mathbf{W}^{1,p}(K)$ for some $p > 2$. The set of

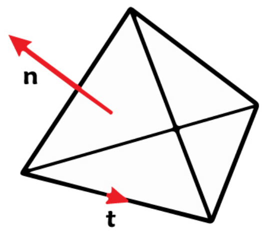


Figure 1.1: A tetrahedron K: $\mathbf{t}$ is an edge tangent vector, $\mathbf{n}$ is a face normal

moments of $\mathbf{u}$ depends on six edges $\mathbf{e}$ of K,

$$M_e(\mathbf{u}) := \left\{ \int_e \mathbf{u} \cdot \mathbf{t}\, q\, de \quad \forall q \in P_{k-1}(e) \right\}, \tag{1.1}$$

on four faces f of K,

$$M_f(\mathbf{u}) := \left\{ \int_f \mathbf{u} \times \mathbf{n} \cdot \mathbf{q}\, ds \quad \forall \mathbf{q} \in [P_{k-2}(f)]^2 \right\}. \tag{1.2}$$

on volume of K,

$$M_K(\mathbf{u}) := \left\{ \int_K \mathbf{u} \cdot \mathbf{q}\, d\mathbf{x} \quad \forall \mathbf{q} \in \mathbf{P}_{k-3}(K) \right\}. \tag{1.3}$$

Remark 1.2 The dofs M_f are given here in the original version of [110], Def. 4. However, from the point of view of affine equivalence it may be useful to use a different, but equivalent representation [106], Remark 5.31.

This set of moments is $\mathbf{R}_k$-unisolvent and curl-conforming as proved in [110], Thm. 1. For any $\mathbf{u} \in \mathbf{W}^{1,p}(K)$, we define a local interpolant $\mathbf{r}_K\mathbf{u} \in \mathbf{R}_k$ such that

$$M_e(\mathbf{u} - \mathbf{r}_K\mathbf{u}) = M_f(\mathbf{u} - \mathbf{r}_K\mathbf{u}) = M_K(\mathbf{u} - \mathbf{u}_K\mathbf{u}) = \{0\}$$

The global interpolant $\mathbf{r}_h\mathbf{u} \in \mathbf{U}_h := \{\mathbf{u} \in \mathbf{H}(\mathrm{curl}\,;\Omega) : \ \mathbf{u}|_K \in \mathbf{R}_k \ \forall K \in \mathcal{T}_h\}$ is defined element-wise:

$$\mathbf{r}_h\mathbf{u}|_K := \mathbf{r}_K(\mathbf{u}|_K) \quad \forall K \in \mathcal{T}_h.$$

The following estimate holds for $\mathbf{r}_h$ ([110], Thm. 2): If $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$ then

$$\|\mathbf{u} - \mathbf{r}_h\mathbf{u}\|_{\mathbf{H}(\mathrm{curl}\,;\Omega)} \le Ch^k\|\mathbf{u}\|_{\mathbf{H}^{k+1}(\Omega)}. \tag{1.4}$$

Finally we set $\mathbf{U}_{0h} := \mathbf{U}_h \cap \mathbf{H}_0(\mathrm{curl}\,;\Omega)$.

Definition 1.3 ($\mathbf{D}_k$-unisolvent and div-conforming dofs) Let K be a tetrahedron in $\mathbb{R}^3$ with faces denoted by f. Let $\mathbf{n}$ be unit outward normal to the faces of tetrahedron. Let $\mathbf{v} \in \mathbf{W}^{1,p}(K)$ for some $p > 2$, and the set of moments of $\mathbf{u}$ depends on four faces f of K,

$$M_f(\mathbf{v}) := \left\{ \int_f \mathbf{v} \cdot \mathbf{n}\, q, ds \quad \forall q \in P_{k-1}(f) \right\}. \tag{1.5}$$

and on volume of K,

$$M_K(\mathbf{v}) := \left\{ \int_K \mathbf{v} \cdot \mathbf{q}\, d\mathbf{x} \quad \forall \mathbf{q} \in \mathbf{P}_{k-2}(K) \right\}. \tag{1.6}$$

This set of moments is $\mathbf{D}_k$-unisolvent and div-conforming as proved in [110], Thm. 3. For any $\mathbf{v} \in \mathbf{W}^{1,p}(K)$, we define a local interpolant $\mathbf{w}_K\mathbf{v} \in \mathbf{D}_k$ as follows:

$$M_f(\mathbf{v} - \mathbf{w}_K\mathbf{v}) = M_K(\mathbf{v} - \mathbf{w}_K\mathbf{v}) = \{0\}.$$

The global interpolant $\mathbf{w}_h\mathbf{v} \in \mathbf{V}_h := \{\mathbf{w} \in \mathbf{H}(\mathrm{div};\Omega) : \ \mathbf{w}|_K \in \mathbf{D}_k \ \forall K \in \mathcal{T}_h\}$ is defined element-wise:

$$\mathbf{w}_h\mathbf{v}|_K := \mathbf{w}_K(\mathbf{v}|_K) \quad \forall K \in \mathcal{T}_h.$$

An error estimate also holds for $\mathbf{w}_h$ (see [100], Eq. (19), and [110], Thm. 4): If $\mathbf{v} \in \mathbf{H}^k(\Omega)$, then

$$\|\mathbf{v} - \mathbf{w}_h\mathbf{v}\| \le Ch^k\|\mathbf{v}\|_{\mathbf{H}^k(\Omega)}. \tag{1.7}$$

1.2.1 Projection Operators for the Lee-Madsen formulation

$\tilde{\mathbf{P}}_h$ be standard $\mathbf{L}^2(\Omega)$ projection into $\mathbf{W}_h$ i.e. for given $\mathbf{w} \in \mathbf{L}^2(\Omega)$, the image $\tilde{\mathbf{P}}_h \mathbf{w} \in \mathbf{W}_h$ is defined as [102, the equation (3.9)]

$$(\tilde{\mathbf{P}}_h \mathbf{w}, \mathbf{\Psi}_h) = (\mathbf{w}, \mathbf{\Psi}_h), \quad \forall \mathbf{\Psi} \in \mathbf{W}_h. \tag{1.8}$$

According to [106, 102], the spaces $\mathbf{W}_h$ and $\mathbf{U}_h$ are related via

$$\nabla \times \mathbf{U}_h \in \mathbf{W}_h. \tag{1.9}$$

If $\mathbf{w} \in \mathbf{H}^k(\Omega)$ then following error bound [102, equation (3.10)] can be derived

$$\|\tilde{\mathbf{P}}_h \mathbf{w} - \mathbf{w}\| \leq C h^k \|\mathbf{u}\|_{\mathbf{H}^k(\Omega)}. \tag{1.10}$$

We will use a second projection operator $\Pi_{1h} : \ \mathbf{H}(\mathrm{curl}\,; \Omega) \to \mathbf{U}_h$ and is defined by

$$\big(\nabla \times \Pi_{1h}\mathbf{u}, \mathbf{\Psi}_h\big) = \big(\nabla \times \mathbf{u}, \mathbf{\Psi}_h\big) \quad \forall \mathbf{\Psi}_h \in \mathbf{W}_h, \tag{1.11}$$
$$(\Pi_{1h}\mathbf{u}, \nabla p_h) = (\mathbf{u}, \nabla p_h) \quad \forall p_h \in S_h^k, \tag{1.12}$$

where S_h^k is defined on $\mathcal{T}_h$ such that

$$S_h^k = \{v_h \in C(\bar{\Omega})/R : \ v_h|_K \in \mathcal{P}_k \ \forall K \in \mathcal{T}_h\},$$

then $\nabla S_h^k \subset \mathbf{U}_h$. An error bound for the projection Π_{1h} can be derived (Theorem 3.2 in [102]). If $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, then

$$\|\mathbf{u} - \Pi_{1h}\mathbf{u}\|_{\mathbf{H}(\mathrm{curl}\,;\Omega)} \leq C h^k \|\mathbf{u}\|_{\mathbf{H}^{k+1}(\Omega)}. \tag{1.13}$$

1.2.2 Projection Operators for the Nédeléc and Raviart-Thomas Formulation

According to [106], and Lemma 5.40 [100], the spaces $\mathbf{U}_{0h}$ and $\mathbf{V}_h$ are related via

$$\nabla \times \mathbf{U}_{0h} \subset \mathbf{V}_h. \tag{1.14}$$

Now the space $\mathbf{V}_h$ is decomposed corresponding to usual $\mathbf{L}^2(\Omega)$ inner product. Let $\nabla \times \mathbf{U}_{0h} = \mathbf{V}_{0h}$ and by the orthogonal decomposition, we can determine the space $\mathbf{V}_{0h}^{\perp}$, We will define the projection operators into the spaces $\mathbf{U}_{0h}$ and $\mathbf{V}_h$, respectively. The projection operator $\Pi_h : \ \mathbf{H}_0(\mathrm{curl}\,; \Omega) \to \mathbf{U}_{0h}$ is defined by

$$\big(\nabla \times \Pi_h\mathbf{u}, \mathbf{\Psi}_h\big) = \big(\nabla \times \mathbf{u}, \mathbf{\Psi}_h\big) \quad \forall \mathbf{\Psi}_h \in \mathbf{V}_h, \tag{1.15}$$
$$(\Pi_h\mathbf{u}, \nabla p_h) = (\mathbf{u}, \nabla p_h) \quad \forall p_h \in S_h^k, \tag{1.16}$$

where S_h^k is the standard space of continuous finite elements on $\mathcal{T}_h$:

$$S_h^k := \{v \in H_0^1(\Omega) : \ v|_K \in \mathcal{P}_k \ \forall K \in \mathcal{T}_h\},$$

see [105], an error bound for the projection Π_h can be be determined derived similar to Π_h (see Theorem 4.5 [102]). If $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, then

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_{\varepsilon_0} \leq Ch^k \|\mathbf{u}\|_{\mathbf{H}^{k+1}(\Omega)}. \tag{1.17}$$

For the partial importance the magnetic field is divergence free vector fields. Therefore, the projection operator for the magnetic field $\mathbf{P}_h$ is defined $\mathbf{L}^2$-orthogonal projection of $\mathbf{H}(\mathrm{div};\Omega)$ onto $\mathbf{V}_{0h} \subset \mathbf{H}_0(\mathrm{div};\Omega)$, we concern

$$(\mathbf{P}_h \mathbf{v}, \boldsymbol{\Phi}_h) = (\mathbf{v}, \boldsymbol{\Phi}_h) \quad \forall \boldsymbol{\Phi}_h \in \mathbf{V}_{0h}. \tag{1.18}$$

By the help of similar arguments as in ([100], eq. 28), the following estimate can be shown for $\mathbf{v} \in \mathbf{H}^k(\Omega)$:

$$\|\mathbf{P}_h \mathbf{v} - \mathbf{v}\|_{\mu_0} \leq Ch^k \|\mathbf{u}\|_{\mathbf{H}^k(\Omega)}. \tag{1.19}$$

Chapter 2

Maxwell's Equation

Let $\Omega \subset \mathbb{R}^3$ be a simply connected domain with a smooth boundary Γ and unit outward normal $\mathbf{n}$. The symbols $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H} = \mathbf{H}(\mathbf{x}, t)$ denote the electric and magnetic field intensities respectively, where the time variable t belongs to some finite interval $(0, T)$, $0 < T < \infty$. Given a current density function $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$, specifying the applied current, Maxwell's equations state that

$$\varepsilon \mathbf{E}_t - \nabla \times \mathbf{H} = \mathbf{J} \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\mu \mathbf{H}_t + \nabla \times \mathbf{E} = 0 \quad \text{in } \Omega \times (0, T). \tag{2.2}$$

The material parameters ε and μ do not depend on time and are piecewise constant. In addition, there exist positive constants $\varepsilon_{\min}$, $\varepsilon_{\max}$, $\mu_{\min}$, $\mu_{\max}$ such that, for all $\mathbf{x} \in \Omega$,

$$\begin{array}{ccccccccc} 0 & < & \varepsilon_{\min} & \leq & \varepsilon(\mathbf{x}) & \leq & \varepsilon_{\max} & < & \infty, \\ 0 & < & \mu_{\min} & \leq & \mu(\mathbf{x}) & \leq & \mu_{\max} & < & \infty. \end{array}$$

A perfect conducting boundary condition on Ω is assumed, that is,

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on } \Gamma \times (0, T).$$

Finally, initial conditions have to be specified so that

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}) \quad \text{and} \quad \mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega, \tag{2.3}$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are given functions on Γ, and $\mathbf{H}_0$ satisfies

$$\nabla \cdot (\mu \mathbf{H}_0) = 0 \quad \text{in } \Omega, \quad \mathbf{H}_0 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma.$$

We assume that the solution $(\mathbf{E}, \mathbf{H})$ of the system (2.1) – (2.3) exists and is unique, for details see [87]. The Maxwell's equations with piecewise constant coefficient have been investigated in [94, 102].

2.1 Weak Formulation

Given $\mathbf{J} \in \mathbf{C}(0,T,\mathbf{L}^2_{\varepsilon^{-1}}(\Omega))$, and the weak solution $(\mathbf{E},\mathbf{H}) \in \big(\mathbf{C}(0,T,\mathbf{H}_0(\mathrm{curl}\,;\Omega)) \cap \mathbf{C}^1(0,T,\mathbf{L}^2_\varepsilon(\Omega))\big) \times \big(\mathbf{C}(0,T,\mathbf{H}(\mathrm{div};\Omega)) \cap \mathbf{C}^1(0,T,\mathbf{L}^2_\mu(\Omega))\big)$ of the system (2.1)–(2.2) satisfies

$$(\varepsilon\mathbf{E}_t,\mathbf{\Psi}) - (\mathbf{H},\nabla\times\mathbf{\Psi}) = (\mathbf{J},\mathbf{\Psi}) \qquad \forall\,\mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}\,;\Omega), \tag{2.4}$$

$$(\mu\mathbf{H}_t,\mathbf{\Phi}) + (\nabla\times\mathbf{E},\mathbf{\Phi}) = 0 \qquad \forall\,\mathbf{\Phi} \in \mathbf{H}(\mathrm{div};\Omega), \tag{2.5}$$

for $t \in (0,T)$ with the initial conditions (2.3).

Theorem 2.1 *Let* $\mathbf{J} \in \mathbf{C}(0,T,\mathbf{L}^2_{\varepsilon^{-1}}(\Omega))$, *and the solution* $(\mathbf{E},\mathbf{H}) \in \big(\mathbf{C}(0,T,\mathbf{H}_0(\mathrm{curl}\,;\Omega)) \cap \mathbf{C}^1(0,T,\mathbf{L}^2_\varepsilon(\Omega))\big) \times \big(\mathbf{C}(0,T,\mathbf{H}(\mathrm{div};\Omega)) \cap \mathbf{C}^1(0,T,\mathbf{L}^2_\mu(\Omega))\big)$ *of the system (2.1)–(2.2) satisfies*

$$\|\mathbf{E}\|_\varepsilon + \|\mathbf{H}\|_\mu \le \sqrt{2}\left[\|\mathbf{E}_0\|_\varepsilon + \|\mathbf{H}_0\|_\mu + \|\mathbf{J}\|_{\mathbf{L}^1\left(0,T,\mathbf{L}^2_{\varepsilon^{-1}}(\Omega)\right)}\right].$$

Proof*: Take test functions* $\mathbf{\Psi} := \mathbf{E}$ *and* $\mathbf{\Phi} := \mathbf{H}$ *in (2.4)–(2.5) respectively, then*

$$(\varepsilon\mathbf{E}_t,\mathbf{E}) - (\mathbf{H},\nabla\times\mathbf{E}) = (\mathbf{J},\mathbf{E}), \tag{2.6}$$

$$(\mu\mathbf{H}_t,\mathbf{H}) + (\nabla\times\mathbf{E},\mathbf{H}) = 0. \tag{2.7}$$

Adding the equations (2.6)–(2.7), we have

$$(\varepsilon\mathbf{E}_t,\mathbf{E}) + (\mu\mathbf{H}_t,\mathbf{H}) = (\mathbf{J},\mathbf{E}),$$

therefore

$$\frac{d}{dt}\Big[\|\mathbf{E}\|^2_\varepsilon + \|\mathbf{H}\|^2_\mu\Big] = 2(\mathbf{J},\mathbf{E}) \le 2\|\mathbf{J}\|_{\varepsilon^{-1}}\|\mathbf{E}\|_\varepsilon.$$

Integrating both sides from 0 *to* T, *we obtain,*

$$\begin{aligned}\|\mathbf{E}\|^2_\varepsilon + \|\mathbf{H}\|^2_\mu &\le \|\mathbf{E}_0\|^2_\varepsilon + \|\mathbf{H}_0\|^2_\mu + 2\int_0^T \|\mathbf{J}\|_{\varepsilon^{-1}}\|\mathbf{E}\|_\varepsilon ds\\ &\le \|\mathbf{E}_0\|^2_\varepsilon + \|\mathbf{H}_0\|^2_\mu + 2\int_0^T \|\mathbf{J}\|_{\varepsilon^{-1}}\sqrt{\|\mathbf{E}\|^2_\varepsilon + \|\mathbf{H}\|^2_\mu}\,ds.\end{aligned}$$

Then it follows from the Gronwall–Ou-Iang's inequality (see, e.g., [114]) that

$$\sqrt{\|\mathbf{E}\|^2_\varepsilon + \|\mathbf{H}\|^2_\mu} \le \sqrt{\|\mathbf{E}_0\|^2_\varepsilon + \|\mathbf{H}_0\|^2_\mu} + \int_0^T \|\mathbf{J}\|_{\varepsilon^{-1}} ds.$$

Since

$$\sqrt{\|\mathbf{E}\|_\varepsilon^2 + \|\mathbf{H}\|_\mu^2} \le \|\mathbf{E}\|_\varepsilon + \|\mathbf{H}\|_\mu \le \sqrt{2}\sqrt{\|\mathbf{E}\|_\varepsilon^2 + \|\mathbf{H}\|_\mu^2},$$

the statement follows. $\square$

Let us now turn to time discretizations for the Maxwell system (2.1)–(2.2). The time interval $(0,T)$ is divided into $N \in \mathbb{N}$ equally spaced subintervals by using nodal points

$$0 =: t^0 < t^1 < t^2 < \cdots < t^N := T,$$

with $t^n = n\Delta t$, $\Delta t > 0$, $n = 0, 1, 2, \cdots, N$.

Replacing the time derivatives in (2.4)–(2.5) at t^n by the backward finite difference quotient, that is

$$\mathbf{E}_t(t^n) \approx \frac{\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})}{\Delta t} \quad \text{etc.},$$

and get a sequence of problems of the type

$$\Big(\varepsilon\frac{\mathbf{E}^n - \mathbf{E}^{n-1}}{\Delta t}, \mathbf{\Psi}\Big) - (\mathbf{H}^n, \nabla\times\mathbf{\Psi}) = (\mathbf{J}^n, \mathbf{\Psi}) \quad \forall\, \mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}\,;\Omega), \tag{2.8}$$

$$\Big(\mu\frac{\mathbf{H}^n - \mathbf{H}^{n-1}}{\Delta t}, \mathbf{\Phi}\Big) + (\nabla\times\mathbf{E}^n, \mathbf{\Phi}) = 0 \quad \forall\, \mathbf{\Phi} \in \mathbf{H}(\mathrm{div};\Omega), \tag{2.9}$$

where $(\mathbf{E}^n, \mathbf{H}^n) \in \mathbf{H}_0(\mathrm{curl}\,;\Omega) \times \mathbf{H}(\mathrm{div};\Omega)$ are to be determined for $n \in \{1, \ldots, N\}$ (as approximations to $(\mathbf{E}(t^n), \mathbf{H}(t^n))$), $(\mathbf{E}^0, \mathbf{H}^0) \in \mathbf{H}_0(\mathrm{curl}\,;\Omega) \times \mathbf{H}(\mathrm{div};\Omega)$ are given (as approximations to $(\mathbf{E}_0, \mathbf{H}_0)$), and $\mathbf{J}^n := \mathbf{J}(t^n) \in \mathbf{L}^2_{\varepsilon^{-1}}(\Omega)$.

Theorem 2.2 *For $0 < \Delta t < 1/2$, there exists a constant $C > 0$ independent of Δt (but dependent on T) such that*

$$\|\mathbf{E}^N\|_\varepsilon^2 + \|\mathbf{H}^N\|_\mu^2 \le C.$$

Proof: In (2.8)–(2.9), we choose the test functions $\mathbf{\Psi} := 2\Delta t\mathbf{E}^n$ and $\mathbf{\Phi} := 2\Delta t\mathbf{H}^n$. Then

$$2(\varepsilon(\mathbf{E}^n - \mathbf{E}^{n-1}), \mathbf{E}^n) - 2\Delta t(\mathbf{H}^n, \nabla\times\mathbf{E}^n) = 2\Delta t(\mathbf{J}^n, \mathbf{E}^n), \tag{2.10}$$

$$2(\mu(\mathbf{H}^n - \mathbf{H}^{n-1}), \mathbf{H}^n) + 2\Delta t(\nabla\times\mathbf{E}^n, \mathbf{H}^n) = 0. \tag{2.11}$$

Adding the equations (2.10) and (2.11), we get

$$2(\varepsilon(\mathbf{E}^n - \mathbf{E}^{n-1}), \mathbf{E}^n) + 2(\mu(\mathbf{H}^n - \mathbf{H}^{n-1}), \mathbf{H}^n = 2\Delta t(\mathbf{J}^n, \mathbf{E}^n).$$

The identity (1) from Lemma 5.1 implies that

$$\begin{aligned}\|\mathbf{E}^n\|_\varepsilon^2 + \|\mathbf{E}^n - \mathbf{E}^{n-1}\|_\varepsilon^2 - \|\mathbf{E}^{n-1}\|_\varepsilon^2 + \|\mathbf{H}^n\|_\mu^2 \\ + \|\mathbf{H}^n - \mathbf{H}^{n-1}\|_\mu^2 - \|\mathbf{H}^{n-1}\|_\mu^2 = 2\Delta t(\mathbf{J}^n, \mathbf{E}^n).\end{aligned} \tag{2.12}$$

The right-hand side is estimated similarly to the proof of the inequality (2) from Lemma 5.1:

$$\begin{aligned}2(\mathbf{J}^n, \mathbf{E}^n) &= 2(\varepsilon^{-1/2}\mathbf{J}^n, \varepsilon^{1/2}\mathbf{E}^n) \\ &\leq \|\varepsilon^{-1/2}\mathbf{J}^n\|^2 + \|\varepsilon^{1/2}\mathbf{E}^n\|^2 = \|\mathbf{J}^n\|_{\varepsilon^{-1}}^2 + \|\mathbf{E}^n\|_\varepsilon^2.\end{aligned}$$

Using this estimate in (2.12), we obtain

$$\|\mathbf{E}^n\|_\varepsilon^2 - \|\mathbf{E}^{n-1}\|_\varepsilon^2 + \|\mathbf{H}^n\|_\mu^2 - \|\mathbf{H}^{n-1}\|_\mu^2 \leq \Delta t\|\mathbf{E}^n\|_\varepsilon^2 + \Delta t\|\mathbf{J}^n\|_{\varepsilon^{-1}}^2.$$

Summing up from $n = 1$ to N, we arrive at

$$\begin{aligned}\|\mathbf{E}^N\|_\varepsilon^2 + \|\mathbf{H}^N\|_\mu^2 &\leq \Delta t\sum_{n=1}^N \|\mathbf{E}^n\|_\varepsilon^2 + \Delta t\sum_{n=1}^N \|\mathbf{J}^n\|_{\varepsilon^{-1}}^2 + \|\mathbf{E}^0\|_\varepsilon^2 + \|\mathbf{H}^0\|_\mu^2 \\ &\leq \Delta t\sum_{n=0}^N \|\mathbf{E}^n\|_\varepsilon^2 + \Delta t\sum_{n=0}^N \|\mathbf{J}^n\|_{\varepsilon^{-1}}^2 + \|\mathbf{E}^0\|_\varepsilon^2 + \|\mathbf{H}^0\|_\mu^2.\end{aligned}$$

Now we are ready to apply Gronwall's inequality (Lemma 5.3) with $\delta := \Delta t \geq 0$, $g_0 := \|\mathbf{E}^0\|_\varepsilon^2 + \|\mathbf{H}^0\|_\mu^2 \geq 0$, $a_n := \|\mathbf{E}^n\|_\varepsilon^2 + \|\mathbf{H}^n\|_\mu^2 \geq 0$, $b_n := 0$, $c_n := \|\mathbf{J}^n\|_{\varepsilon^{-1}}^2 \geq 0$, and $\gamma_n := 1 \geq 0$. Then the condition $\gamma_n\delta < 1$ corresponds to $\Delta t < 1$ and the final estimate follows from the observation that $(n+1)\Delta t \leq T + \Delta t \leq T + 1/2$:

$$\begin{aligned}\|\mathbf{E}^N\|_\varepsilon^2 + \|\mathbf{H}^N\|_\mu^2 &\leq \left(\Delta t\sum_{n=0}^N \|\mathbf{J}^n\|_{\varepsilon^{-1}}^2 + \|\mathbf{E}^0\|_\varepsilon^2 + \|\mathbf{H}^0\|_\mu^2\right) \\ &\quad\times \exp\left(\Delta t\sum_{n=0}^N (1-\Delta t)^{-1}\right) \\ &\leq \left(\sum_{n=0}^N \|\mathbf{J}^n\|_{\varepsilon^{-1}}^2\Delta t + \|\mathbf{E}^0\|_\varepsilon^2 + \|\mathbf{H}^0\|_\mu^2\right)\exp\left(2T+1\right).\end{aligned}$$

It remains to note that the term $\sum_{n=0}^N \|\mathbf{J}^n\|_{\varepsilon^{-1}}^2\Delta t$ is an approximation to $\int_0^T \|\mathbf{J}(t)\|_{\varepsilon^{-1}}^2 dt = \|\mathbf{J}\|_{\mathbf{L}^2(0,T,\mathbf{L}_{\varepsilon^{-1}}^2(\Omega))}^2$ and thus bounded. $\square$

Next we want to prove an estimate of the error in time. To do so, we introduce the errors

$$\zeta_c^n := \mathbf{E}(t^n) - \mathbf{E}^n, \quad \xi_c^n := \mathbf{H}(t^n) - \mathbf{H}^n. \tag{2.13}$$

Theorem 2.3 *If* $(\mathbf{E},\mathbf{H}) \in \big(\mathbf{C}(0,T,\mathbf{H}_0(\mathrm{curl}\,;\Omega)) \cap \mathbf{C}^2(0,T,\mathbf{L}^2_\varepsilon(\Omega))\big) \times \big(\mathbf{C}(0,T,\mathbf{H}(\mathrm{div};\Omega)) \cap \mathbf{C}^2(0,T,\mathbf{L}^2_\mu(\Omega))\big)$ *and if the time step* Δt *is sufficiently small, then there exists a constant* $C > 0$ *independent of* Δt *(but dependent on* T*) such that*

$$\|\zeta_c^N\|_\varepsilon^2 + \|\xi_c^N\|_\mu^2 \le C\big(\Delta t + \|\zeta_c^0\|_\varepsilon + \|\xi_c^0\|_\mu\big).$$

Proof: From Taylor's formula with integral remainder it follows that

$$\mathbf{E}(t) = \mathbf{E}(t^n) + \mathbf{E}_t(t^n)(t-t^n) + \int_{t^n}^{t}(t-s)\mathbf{E}_{tt}(s)ds,$$

hence

$$\Big(\varepsilon\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}, \mathbf{\Psi}\Big) = (\varepsilon\mathbf{E}_t(t^n), \mathbf{\Psi}) + (\varepsilon\mathbf{R}^n_{\mathbf{E}}, \mathbf{\Psi}), \tag{2.14}$$

where

$$\mathbf{R}^n_{\mathbf{E}} := \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}(t^{n-1}-s)\mathbf{E}_{tt}(s)ds.$$

An analogous relation can be obtained with respect to $\mathbf{H}$. Making use of (2.4)–(2.5), we get

$$\begin{aligned}
&\Big(\varepsilon\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}, \mathbf{\Psi}\Big) - (\mathbf{H}(t^n), \nabla\times\mathbf{\Psi}) \\
&= (\mathbf{J}^n + \varepsilon\mathbf{R}^n_{\mathbf{E}}, \mathbf{\Psi}) \quad \forall\,\mathbf{\Psi}\in\mathbf{H}_0(\mathrm{curl}\,;\Omega), && (2.15)\\
&\Big(\mu\frac{\mathbf{H}(t^n)-\mathbf{H}(t^{n-1})}{\Delta t}, \mathbf{\Phi}\Big) + (\nabla\times\mathbf{E}(t^n), \mathbf{\Phi}) = (\mu\mathbf{R}^n_{\mathbf{H}}, \mathbf{\Phi}) \quad \forall\,\mathbf{\Phi}\in\mathbf{H}(\mathrm{div};\Omega). && (2.16)
\end{aligned}$$

Now the equations (2.8)-(2.9) are subtracted from the equations (2.15)-(2.16). Together with (2.13) the result is:

$$\Big(\varepsilon\frac{\zeta_c^n-\zeta_c^{n-1}}{\Delta t}, \mathbf{\Psi}\Big) - \big(\xi_c^n, \nabla\times\mathbf{\Psi}\big) = (\varepsilon\mathbf{R}^n_{\mathbf{E}}, \mathbf{\Psi}), \tag{2.17}$$

$$\Big(\mu\frac{\xi_c^n-\xi_c^{n-1}}{\Delta t}, \mathbf{\Phi}\Big) + \big(\nabla\times\zeta_c^n, \mathbf{\Phi}\big) = (\mu\mathbf{R}^n_{\mathbf{H}}, \mathbf{\Phi}). \tag{2.18}$$

Taking $\mathbf{\Psi} := 2\Delta t \zeta_c^n$ and $\mathbf{\Phi} := 2\Delta t \xi_c^n$ in equations (2.17) and (2.18), by the same reasoning as in the proof of Theorem 2.2, from this we obtain the estimate

$$\begin{aligned}
&\|\zeta_c^n\|_\varepsilon^2 - \|\zeta_c^{n-1}\|_\varepsilon^2 + \|\xi_c^n\|_\mu^2 - \|\xi_c^{n-1}\|_\mu^2 \\
&\qquad\qquad \leq \Delta t\big[\|\zeta_c^n\|_\varepsilon^2 + \|\xi_c^n\|_\mu^2\big] + \Delta t\left[\|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 + \|\mathbf{R}_{\mathbf{H}}^n\|_\mu^2\right].
\end{aligned}$$

The summation leads to

$$\begin{aligned}
\|\zeta_c^N\|_\varepsilon^2 + \|\xi_c^N\|_\mu^2 \leq \Big[&\Delta t \sum_{n=1}^{N} \left[\|\zeta_c^n\|_\varepsilon^2 + \|\xi_c^n\|_\mu^2\right] + \Delta t \sum_{n=1}^{N} \left[\|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 + \|\mathbf{R}_{\mathbf{H}}^n\|_\mu^2\right] \\
&+ \|\zeta_c^0\|_\varepsilon^2 + \|\xi_c^0\|_\mu^2\Big].
\end{aligned}$$

Next we apply Gronwall's inequality (Lemma 5.3) with $\delta := \Delta t \geq 0$, $g_0 := \|\zeta_c^0\|_\varepsilon^2 + \|\xi_c^0\|_\mu^2 \geq 0$, $a_n := \|\zeta_c^n\|_\varepsilon^2 + \|\xi_c^n\|_\mu^2 \geq 0$, $b_n := 0$, $c_n := \|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 + \|\mathbf{R}_{\mathbf{H}}^n\|_\mu^2 \geq 0$ $(n \geq 1)$, $c_0 := 0$, and $\gamma_n := 1 \geq 0$. Then the condition $\gamma_n \delta < 1$ corresponds to $\Delta t < 1$ and we get

$$\begin{aligned}
\|\zeta_c^N\|_\varepsilon^2 + \|\xi_c^N\|_\mu^2 &\leq \Big[\big(\Delta t \sum_{n=0}^{N} \left[\|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 + \|\mathbf{R}_{\mathbf{H}}^n\|_\mu^2\right] + \|\zeta_c^0\|_\varepsilon^2 + \|\xi_c^0\|_\mu^2\big) \\
&\qquad \times \exp\big(\Delta t \sum_{n=0}^{N} (1 - \Delta t)^{-1}\big)\Big] \\
&\leq \Big[\big(\sum_{n=0}^{N} \left[\|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 + \|\mathbf{R}_{\mathbf{H}}^n\|_\mu^2\right] \Delta t + \|\zeta_c^0\|_\varepsilon^2 + \|\xi_c^0\|_\mu^2\big) \\
&\qquad \times \exp\left(2T + 1\right)\Big].
\end{aligned}$$

It remains to estimate the sum terms. Since

$$\begin{aligned}
\|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 &= \frac{1}{(\Delta t)^2} \left\| \int_{t^{n-1}}^{t^n} (t^{n-1} - s)\mathbf{E}_{tt}(s) ds. \right\|_\varepsilon^2 \\
&\leq \frac{1}{(\Delta t)^2} \int_{t^{n-1}}^{t^n} (s - t^{n-1})^2 ds \int_{t^{n-1}}^{t^n} \|\mathbf{E}_{tt}(s)\|_\varepsilon^2 ds \\
&= \frac{\Delta t}{3} \int_{t^{n-1}}^{t^n} \|\mathbf{E}_{tt}(s)\|_\varepsilon^2 ds\,,
\end{aligned}$$

it follows that

$$\sum_{n=0}^{N} \|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 \Delta t \leq \frac{(\Delta t)^2}{3} \int_0^T \|\mathbf{E}_{tt}(s)\|_\varepsilon^2 ds = \frac{(\Delta t)^2}{3} \|\mathbf{E}_{tt}\|_{\mathbf{L}^2(0,T,\mathbf{L}_\varepsilon^2(\Omega))}^2.$$

In summary, we get

$$\|\zeta_c^N\|_\varepsilon^2 + \|\xi_c^N\|_\mu^2 \leq \Big(\frac{(\Delta t)^2}{3}\big[\|\mathbf{E}_{tt}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_\varepsilon(\Omega))} + \|\mathbf{H}_{tt}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_\mu(\Omega))}\big] + \|\zeta_c^0\|_\varepsilon^2 + \|\xi_c^0\|_\mu^2\Big)\exp\left(2T+1\right).$$

$\square$

2.2 Full Discretization Using the Backward Euler Method

The fully discrete electric and magnetic fields $(\mathbf{E}_h^n, \mathbf{H}_h^n) \in \mathbf{U}_{0h} \times \mathbf{V}_h$ satisfy

$$\Big(\varepsilon \frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\Delta t}, \mathbf{\Psi}_h\Big) - (\mathbf{H}_h^n, \nabla \times \mathbf{\Psi}_h) = (\mathbf{J}^n, \mathbf{\Psi}_h) \qquad \forall\, \mathbf{\Psi}_h \in \mathbf{U}_{0h}, \tag{2.19}$$

$$\Big(\mu \frac{\mathbf{H}_h^n - \mathbf{H}_h^{n-1}}{\Delta t}, \mathbf{\Phi}_h\Big) + (\nabla \times \mathbf{E}_h^n, \mathbf{\Phi}_h) = 0 \qquad \forall\, \Phi_h \in \mathbf{V}_h, \tag{2.20}$$

where $(\mathbf{E}_h^n, \mathbf{H}_h^n) \in \mathbf{U}_{0h} \times \mathbf{V}_h$ are to be determined for $n \in \{1, \ldots, N\}$ and $(\mathbf{E}_h^0, \mathbf{H}_h^0) \in \mathbf{U}_{0h} \times \mathbf{V}_h$ are given (and will be specified later). Before formulating the theorem, we still introduce a few terms. The full error of the electric field will be denoted by

$$\zeta^n := \mathbf{E}(t^n) - \mathbf{E}_h^n = \eta^n - \eta_h^n, \tag{2.21}$$

where

$$\eta^n := \mathbf{E}(t^n) - \Pi_h \mathbf{E}(t^n), \quad \eta_h^n := \mathbf{E}_h^n - \Pi_h \mathbf{E}(t^n). \tag{2.22}$$

Similarly for the magnetic field:

$$\xi^n := \mathbf{H}(t^n) - \mathbf{H}_h^n = \theta^n - \theta_h^n \tag{2.23}$$

with

$$\theta^n := \mathbf{H}(t^n) - \mathbf{P}_h \mathbf{H}(t^n), \quad \theta_h^n := \mathbf{H}_h^n - \mathbf{P}_h \mathbf{H}(t^n). \tag{2.24}$$

Theorem 2.4 *Let* $(\mathbf{E}, \mathbf{H})$ *be the solution of (2.4)–(2.5) such that, for some* $k \in \mathbb{N}$,

$$\begin{aligned} &\mathbf{E} \in C(0, T, \mathbf{H}_0(\mathrm{curl}\,; \Omega)) \cap C^2(0, T, \mathbf{L}^2_\varepsilon(\Omega) \cap \mathbf{H}^{k+1}(\Omega)), \\ &\mathbf{H} \in C^2(0, T, \mathbf{L}^2_\mu(\Omega) \cap \mathbf{H}^k(\Omega)), \end{aligned}$$

and $(\mathbf{E}_h^n, \mathbf{H}_h^n)$ *be the fully discrete solution of (2.19)-(2.20). Then, for sufficiently small* Δt *and* h*, the following error estimate holds:*

$$\|\zeta^N\|_\varepsilon + \|\xi^N\|_\mu \le C\big[\Delta t + h^k + h^k \Delta t\big],$$

where the constant $C > 0$ *does not depend on* Δt *and* h *(the structure of* C *will be seen from the proof).*

Proof: Taking $\mathbf{\Psi} = \mathbf{\Psi}_h$ and $\mathbf{\Phi} = \mathbf{\Phi}_h$ in (2.15)-(2.16), subtracting the system (2.19)-(2.20) from the system (2.4)–(2.5) and using the definitions (2.21), (2.23), we obtain:

$$\Big(\varepsilon\frac{\zeta^n - \zeta^{n-1}}{\Delta t}, \mathbf{\Psi}_h\Big) - (\xi^n, \nabla \times \mathbf{\Psi}_h) = (\varepsilon \mathbf{R}^n_{\mathbf{E}}, \mathbf{\Psi}_h),$$
$$\Big(\mu\frac{\xi^n - \xi^{n-1}}{\Delta t}, \mathbf{\Phi}_h\Big) + (\nabla \times \zeta^n, \mathbf{\Phi}_h) = (\mu \mathbf{R}^n_{\mathbf{H}}, \mathbf{\Phi}_h).$$

Furthermore, using the decompositions (2.22), (2.24), after a simple rearrangement we arrive at

$$\Big(\varepsilon\frac{(\eta^n - \eta^{n-1}) - (\eta^n_h - \eta^{n-1}_h)}{\Delta t}, \mathbf{\Psi}_h\Big) - (\theta^n - \theta^n_h, \nabla \times \mathbf{\Psi}_h) = (\varepsilon \mathbf{R}^n_{\mathbf{E}}, \mathbf{\Psi}_h),$$
$$\Big(\mu\frac{(\theta^n - \theta^{n-1}) - (\theta^n_h - \theta^{n-1}_h)}{\Delta t}, \mathbf{\Phi}_h\Big) + (\nabla \times (\eta^n - \eta^n_h), \mathbf{\Phi}_h) = (\mu \mathbf{R}^n_{\mathbf{H}}, \mathbf{\Phi}_h).$$

Now we set $\mathbf{\Psi}_h := -2\Delta t \eta^n_h$ and $\mathbf{\Phi}_h := -2\Delta t \theta^n_h$:

$$\begin{aligned}
2\big(\varepsilon(\eta^n_h - \eta^{n-1}_h), \eta^n_h\big) - 2\Delta t(\theta^n_h, \nabla \times \eta^n_h &= 2\big(\varepsilon(\eta^n - \eta^{n-1}), \eta^n_h\big) \\
&\quad - 2\Delta t(\theta^n, \nabla \times \eta^n_h) - 2\Delta t(\varepsilon \mathbf{R}^n_{\mathbf{E}}, \eta^n_h), \\
2\big(\mu(\theta^n_h - \theta^{n-1}_h), \theta^n_h\big) + 2\Delta t\big(\nabla \times \eta^n_h, \theta^n_h\big) &= 2\big(\mu(\theta^n - \theta^{n-1}), \theta^n_h\big) \\
&\quad + 2\Delta t\big(\nabla \times \eta^n, \theta^n_h\big) - 2\Delta t(\mu \mathbf{R}^n_{\mathbf{H}}, \theta^n_h).
\end{aligned}$$

Thanks to (1.15), (1.18) and (1.14), the middle terms on the right-hand sides vanish. Adding the resulting equations, we get

$$\begin{aligned}
&2\big(\varepsilon(\eta^n_h - \eta^{n-1}_h), \eta^n_h\big) + 2\big(\mu(\theta^n_h - \theta^{n-1}_h), \theta^n_h\big) \\
&= 2\big(\varepsilon(\eta^n - \eta^{n-1}), \eta^n_h\big) + 2\big(\mu(\theta^n - \theta^{n-1}), \theta^n_h\big) \\
&- 2\Delta t(\varepsilon \mathbf{R}^n_{\mathbf{E}}, \eta^n_h) - 2\Delta t(\mu \mathbf{R}^n_{\mathbf{H}}, \theta^n_h).
\end{aligned}$$

Now the identity (1) from Lemma 5.1 allows to rewrite the left-hand side as,

$$\begin{aligned}
\|\eta^n_h\|^2_\varepsilon + \|\eta^n_h - \eta^{n-1}_h\|^2_\varepsilon - \|\eta^{n-1}_h\|^2_\varepsilon + \|\theta^n_h\|^2_\mu + \|\theta^n_h - \theta^{n-1}_h\|^2_\mu - \|\theta^{n-1}_h\|^2_\mu \\
= 2\big(\varepsilon(\eta^n - \eta^{n-1}), \eta^n_h\big) + 2\big(\mu(\theta^n - \theta^{n-1}), \theta^n_h\big) \\
- 2\Delta t(\varepsilon \mathbf{R}^n_{\mathbf{E}}, \eta^n_h) - 2\Delta t(\mu \mathbf{R}^n_{\mathbf{H}}, \theta^n_h).
\end{aligned} \tag{2.25}$$

The first two terms on the right-hand side are treated by means of formula (2.14). Namely, replacing there $\mathbf{E}(t^n)$ by $(\mathbf{I} - \Pi_h)\mathbf{E}(t^n)$, we obtain

$$\big(\varepsilon(\eta^n - \eta^{n-1}), \eta^n_h\big) = \Delta t(\varepsilon(\mathbf{I} - \Pi_h)\mathbf{E}_t(t^n), \eta^n_h) + \Delta t(\varepsilon(\mathbf{I} - \Pi_h)\mathbf{R}^n_{\mathbf{E}}, \eta^n_h).$$

In the same way we get

$$\big(\mu(\theta^n - \theta^{n-1}), \theta_h^n\big) = \Delta t(\mu(\mathbf{I} - \mathbf{P}_h)\mathbf{H}_t(t^n), \theta_h^n) + \Delta t(\mu(\mathbf{I} - \mathbf{P}_h)\mathbf{R}_{\mathbf{H}}^n, \theta_h^n).$$

Using this in (2.25) and applying the estimate (2) from Lemma 5.1 to each of the resulting terms on the right-hand side, we obtain

$$\begin{aligned}
&\|\eta_h^n\|_\varepsilon^2 + \|\eta_h^n - \eta_h^{n-1}\|_\varepsilon^2 - \|\eta_h^{n-1}\|_\varepsilon^2 + \|\theta_h^n\|_\mu^2 + \|\theta_h^n - \theta_h^{n-1}\|_\mu^2 - \|\theta_h^{n-1}\|_\mu^2 \\
&\le \Delta t\|(\mathbf{I} - \Pi_h)\mathbf{E}_t(t^n)\|_\varepsilon^2 + \Delta t\|(\mathbf{I} - \Pi_h)\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 + \Delta t\|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 \\
&+ 3\Delta t\|\eta_h^n\|_\varepsilon^2 + \Delta t\|(\mathbf{I} - \mathbf{P}_h)\mathbf{H}_t(t^n)\|_\mu^2 + \Delta t\|(\mathbf{I} - \mathbf{P}_h)\mathbf{R}_{\mathbf{H}}^n\|_\mu^2 \\
&+ \Delta t\|\mathbf{R}_{\mathbf{H}}^n\|_\mu^2 + 3\Delta t\|\theta_h^n\|_\mu^2
\end{aligned}$$

Summing up from $n = 1$ to N and ignoring the second and fifth terms on the left-hand side, we arrive at

$$\begin{aligned}
\|\eta_h^N\|_\varepsilon^2 + \|\theta_h^N\|_\mu^2 &\le 3\Delta t \sum_{n=1}^{N} \left[\|\eta_h^n\|_\varepsilon^2 + \|\theta_h^n\|_\mu^2\right] \\
&+ \Delta t \sum_{n=1}^{N} \Big[\|(\mathbf{I} - \Pi_h)\mathbf{E}_t(t^n)\|_\varepsilon^2 + \|(\mathbf{I} - \Pi_h)\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 + \|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 \\
&+ \|(\mathbf{I} - \mathbf{P}_h)\mathbf{H}_t(t^n)\|_\mu^2 + \|(\mathbf{I} - \mathbf{P}_h)\mathbf{R}_{\mathbf{H}}^n\|_\mu^2 + \|\mathbf{R}_{\mathbf{H}}^n\|_\mu^2\Big] \\
&+ \|\eta_h^0\|_\varepsilon^2 + \|\theta_h^0\|_\mu^2.
\end{aligned}$$

Now we are ready to apply Gronwall's inequality (Lemma 5.3) with $\delta := \Delta t \ge 0$, $g_0 := \|\eta_h^0\|_\varepsilon^2 + \|\theta_h^0\|_\mu^2 \ge 0$, $a_n := \|\eta_h^n\|_\varepsilon^2 + \|\theta_h^n\|_\mu^2 \ge 0$, $b_n := 0$, $c_n := \|(\mathbf{I} - \Pi_h)\mathbf{E}_t(t^n)\|_\varepsilon^2 + \|(\mathbf{I} - \Pi_h)\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 + \|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 + \|(\mathbf{I} - \mathbf{P}_h)\mathbf{H}_t(t^n)\|_\mu^2 + \|(\mathbf{I} - \mathbf{P}_h)\mathbf{R}_{\mathbf{H}}^n\|_\mu^2 + \|\mathbf{R}_{\mathbf{H}}^n\|_\mu^2 \ge 0$, and $\gamma_n := 3 \ge 0$. Then the condition $\gamma_n\delta < 1$ corresponds to $\Delta t < 1/3$ and we obtain, say for $\Delta t < 1/6$,

$$\begin{aligned}
\|\eta_h^N\|_\varepsilon^2 + \|\theta_h^N\|_\mu^2 \le \Bigg(\Delta t \sum_{n=0}^{N} \Big[&\|(\mathbf{I} - \Pi_h)\mathbf{E}_t(t^n)\|_\varepsilon^2 + \|(\mathbf{I} - \Pi_h)\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 \\
&+ \|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 + \|(\mathbf{I} - \mathbf{P}_h)\mathbf{H}_t(t^n)\|_\mu^2 + \|(\mathbf{I} - \mathbf{P}_h)\mathbf{R}_{\mathbf{H}}^n\|_\mu^2 \\
&+ \|\mathbf{R}_{\mathbf{H}}^n\|_\mu^2\Big] + \|\eta_h^0\|_\varepsilon^2 + \|\theta_h^0\|_\mu^2\Bigg) \exp(6T + 1).
\end{aligned}$$

From the end of the proof of Thm. 2.3 it is known that

$$\Delta t \sum_{n=0}^{N} \left[\|\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2 + \|\mathbf{R}_{\mathbf{H}}^n\|_\mu^2\right] \le \frac{(\Delta t)^2}{3}\left[\|\mathbf{E}_{tt}\|_{\mathbf{L}^2(0,T,\mathbf{L}_\varepsilon^2(\Omega))}^2 + \|\mathbf{H}_{tt}\|_{\mathbf{L}^2(0,T,\mathbf{L}_\mu^2(\Omega))}^2\right].$$

Furthermore, from (1.17) we see that,

$$\|(\mathbf{I}-\Pi_h)\mathbf{E}_t(t^n)\|_\varepsilon \le Ch^k\|\mathbf{E}_t(t^n)\|_{\mathbf{H}^{k+1}(\Omega)},$$
$$\|(\mathbf{I}-\Pi_h)\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon \le Ch^k\|\mathbf{R}_{\mathbf{E}}^n\|_{\mathbf{H}^{k+1}(\Omega)},$$

and (1.19) yields the estimates,

$$\|(\mathbf{I}-\mathbf{P}_h)\mathbf{H}_t(t^n)\|_\mu \le Ch^k\|\mathbf{H}_t(t^n)\|_{\mathbf{H}^{k}(\Omega)}$$
$$\|(\mathbf{I}-\mathbf{P}_h)\mathbf{R}_{\mathbf{H}}^n\|_\mu \le Ch^k\|\mathbf{R}_{\mathbf{H}}^n\|_{\mathbf{H}^{k}(\Omega)}.$$

Therefore,

$$\begin{aligned}
&\Delta t\sum_{n=0}^{N}\big[\|(\mathbf{I}-\Pi_h)\mathbf{E}_t(t^n)\|_\varepsilon^2+\|(\mathbf{I}-\mathbf{P}_h)\mathbf{H}_t(t^n)\|_\mu^2\big]\\
&\le Ch^{2k}\sum_{n=0}^{N}\big[\|\mathbf{E}_t(t^n)\|_{\mathbf{H}^{k+1}(\Omega)}^2+\|\mathbf{H}_t(t^n)\|_{\mathbf{H}^{k}(\Omega)}^2\big]\Delta t\\
&\le Ch^{2k}\int_0^T\big[\|\mathbf{E}_t(t)\|_{\mathbf{H}^{k+1}(\Omega)}^2+\|\mathbf{H}_t(t)\|_{\mathbf{H}^{k}(\Omega)}^2\big]dt\\
&= Ch^{2k}\big[\|\mathbf{E}_t\|_{\mathbf{L}^2(0,T,\mathbf{H}^{k+1}(\Omega))}^2+\|\mathbf{H}_t\|_{\mathbf{L}^2(0,T,\mathbf{H}^{k}(\Omega))}^2\big],
\end{aligned}$$

and

$$\begin{aligned}
&\Delta t\sum_{n=0}^{N}\big[\|(\mathbf{I}-\Pi_h)\mathbf{R}_{\mathbf{E}}^n\|_\varepsilon^2+\|(\mathbf{I}-\mathbf{P}_h)\mathbf{R}_{\mathbf{H}}^n\|_\mu^2\big]\\
&\le Ch^{2k}\sum_{n=0}^{N}\big[\|\mathbf{R}_{\mathbf{E}}^n\|_{\mathbf{H}^{k+1}(\Omega)}^2+\|\mathbf{R}_{\mathbf{H}}^n\|_{\mathbf{H}^{k}(\Omega)}^2\big]\Delta t\\
&\le Ch^{2k}(\Delta t)^2\big[\|\mathbf{E}_{tt}\|_{\mathbf{L}^2(0,T,\mathbf{H}^{k+1}(\Omega))}^2+\|\mathbf{H}_{tt}\|_{\mathbf{L}^2(0,T,\mathbf{H}^{k}(\Omega))}^2\big].
\end{aligned}$$

Finally, if we take

$$\mathbf{E}_h^0 := \Pi_h\mathbf{E}_0, \quad \mathbf{H}_h^0 := \mathbf{P}_h\mathbf{H}_0, \tag{2.26}$$

we conclude that,

$$\|\eta_h^N\|_\varepsilon+\|\theta_h^N\|_\mu \le C\big[\Delta t+h^k+h^k\Delta t\big].$$

The terms $\|\eta^N\|_\varepsilon$ and $\|\theta^N\|_\mu$ are estimated by (1.17) and (1.19), respectively:

$$\|\eta^N\|_\varepsilon = \|(\mathbf{I}-\Pi_h)\mathbf{E}(t^N)\|_\varepsilon \le Ch^k\|\mathbf{E}(t^N)\|_{\mathbf{H}^{k+1}(\Omega)},$$
$$\|\theta^N\|_\mu = \|(\mathbf{I}-\mathbf{P}_h)\mathbf{H}(t^N)\|_\mu \le Ch^k\|\mathbf{H}(t^N)\|_{\mathbf{H}^{k}(\Omega)}.$$

The triangle inequality yields the stated result:

$$\begin{aligned}\|\mathbf{E}(t^N) - \mathbf{E}_h^N\|_\varepsilon + \|\mathbf{H}(t^N) - \mathbf{H}_h^N\|_\mu &\le \|\eta^N\|_\varepsilon + \|\eta_h^N\|_\varepsilon + \|\theta^N\|_\mu + \|\theta_h^N\|_\mu \\ &\le C\big[\Delta t + h^k + h^k \Delta t\big].\end{aligned}$$

$\square$

2.3 Numerical Results

Denoting by $\mathbf{e}^n$, $\mathbf{h}^n$ and $\mathbf{j}^n$ the representation vectors of $\mathbf{E}_h^n$, $\mathbf{H}_h^n$ and the $\mathbf{L}^2$-orthogonal projection of $\mathbf{J}^n$ onto $\mathbf{U}_{0h}$, the method (2.19)–(2.20) can be written as follows:

$$\mathbf{M}_\varepsilon \frac{\mathbf{e}^n - \mathbf{e}^{n-1}}{\Delta t} = \mathbf{G}^\top \mathbf{h}^n + \mathbf{j}^n, \tag{2.27}$$

$$\mathbf{M}_\mu \frac{\mathbf{h}^n - \mathbf{h}^{n-1}}{\Delta t} = -\mathbf{G}\mathbf{e}^n, \tag{2.28}$$

where $\mathbf{M}_\varepsilon$ is the positively definite mass matrix of size $\dim \mathbf{U}_{0h} \times \dim \mathbf{U}_{0h}$ for the material parameter ε, $\mathbf{M}_\mu$ is the positively definite mass matrix of size $\dim \mathbf{V}_h \times \dim \mathbf{V}_h$ for the material parameter μ, and $\mathbf{G}$ is a discrete representation of $-\operatorname{curl}$ with size $\dim \mathbf{V}_h \times \dim \mathbf{U}_{0h}$.

2.3.1 Implementation of the Backward Euler Method

The formal algorithm for the system (2.27)-(2.28) reads as follows:

Compute the number of time steps:

$nstep := \dfrac{T - t_0}{\Delta t}$

Compute the initial values for the electric and magnetic fields:

$\mathbf{e}^0 \leftarrow \mathbf{E}_0$

$\mathbf{h}^0 \leftarrow \mathbf{H}_0$

Loop over time steps:

`for` $n = 1$ `to` $nstep$ `do:`

Begin integration method update:

$\mathbf{e}_{in} \leftarrow \mathbf{e}^{n-1}$

$\mathbf{h}_{in} \leftarrow \mathbf{h}^{n-1}$

Update the electric and magnetic fields values:

$\mathbf{e}_{out} \leftarrow \mathbf{e}_{in} + \Delta t \mathbf{M}_\varepsilon^{-1}\big(\mathbf{G}^\top \mathbf{h}_{out} + \mathbf{j}^n\big)$

$\mathbf{h}_{out} \leftarrow \mathbf{h}_{in} + \Delta t \mathbf{M}_\mu^{-1} \mathbf{G}\mathbf{e}_{out}$

Update the electric and magnetic fields values for this time step:

$\mathbf{e}^n \leftarrow \mathbf{e}_{out}$

$\mathbf{h}^n \leftarrow \mathbf{h}_{out}$

`end for`

Completion:

$\mathbf{e}^N \leftarrow \mathbf{e}^{nstep}$

$\mathbf{h}^N \leftarrow \mathbf{h}^{nstep}$

2.3.2 Implementation of the Symplectic Method

We have also tested the application of symplectic time integration methods. A 4th order symplectic integration algorithm for the time dependent Maxwell's equations with parameters

$$\begin{aligned}
\beta_1 &= \frac{2+2^{\frac{1}{3}}+2^{-\frac{1}{3}}}{6}, & \alpha_1 &= 0,\\
\beta_2 &= \frac{1-2^{\frac{1}{3}}-2^{-\frac{1}{3}}}{6}, & \alpha_2 &= \frac{1}{2-2^{\frac{1}{3}}},\\
\beta_3 &= \frac{1-2^{\frac{1}{3}}-2^{-\frac{1}{3}}}{6}, & \alpha_3 &= \frac{1}{1-2^{\frac{2}{3}}},\\
\beta_4 &= \frac{2+2^{\frac{1}{3}}+2^{-\frac{1}{3}}}{6}, & \alpha_4 &= \frac{1}{2-2^{\frac{1}{3}}}
\end{aligned}$$

(see [48], [119]) is given as follows:

Compute the number of time steps:

$$nstep := \frac{T-t_0}{\Delta t}$$

Compute the initial values for the electric and magnetic fields:

$\mathbf{e}^0 \leftarrow \mathbf{E}_0$

$\mathbf{h}^0 \leftarrow \mathbf{H}_0$

Loop over time steps:

`for` $n = 1$ `to` $nstep$ `do:`

Begin integration method update:

$\mathbf{e}_{in} \leftarrow \mathbf{e}^{n-1}$

$\mathbf{h}_{in} \leftarrow \mathbf{h}^{n-1}$

Update the electric and magnetic fields values:

```
for j = 1 to 4 do
```
$\mathbf{e}_{out} \leftarrow \mathbf{e}_{in} + \alpha_j \Delta t \mathbf{M}_{\varepsilon}^{-1}\big(\mathbf{G}^{\top}\mathbf{h}_{in} + \mathbf{j}^n\big)$

$\mathbf{h}_{out} \leftarrow \mathbf{h}_{in} + \beta_j \Delta t \mathbf{M}_{\mu}^{-1}\mathbf{G}\mathbf{e}_{out}$

$\mathbf{e}_{in} \leftarrow \mathbf{e}_{out}$

$\mathbf{h}_{in} \leftarrow \mathbf{h}_{out}$

```
end for
```
Update the electric and magnetic fields values for this time step:

$\mathbf{e}^n \leftarrow \mathbf{e}_{out}$

$\mathbf{h}^n \leftarrow \mathbf{h}_{out}$

```
end for
```
Completion:

$\mathbf{e}^N \leftarrow \mathbf{e}^{nstep}$

$\mathbf{h}^N \leftarrow \mathbf{h}^{nstep}$

Krylov solvers were used to invert the mass matrices $\mathbf{M}_\varepsilon$ and $\mathbf{M}_\mu$. A preconditioned conjugate gradient solver was also implemented.

2.3.3 Energy Energy Conservation

In order to verify a correct physical behaviour of the numerical methods, we considered the energy evolution, too. The discrete instantaneous energy is the total energy that is stored in the discrete electric and magnetics fields. It is computed as

$$\mathbf{Energy} = \frac{1}{2}\big(\mathbf{e}^T\mathbf{M}_\varepsilon\mathbf{e} + \mathbf{h}^T\mathbf{M}_\mu\mathbf{h}\big).$$

It is an substantial advantage of the symplectic method that it conserves the energy, see Fig. 2.1.

2.3.4 Simulation Results, Validations and Discussion

A number of numerical experiments were performed to approximate solutions of time dependent Maxwell's problems. We visualized the electromagnetic fields for cases where the exact solution is known, but also for cases with unknown analytical solution, and checked the stability and convergence properties in problems with complicated geometries. The main object of these simulations was to validate the code. The simulations are conditionally stable in the case of the symplectic time integration method.

Example 2.5 This test example is characterized by the following parameters, where a symplectic time integration method is applied to a Fichera mesh. The frequency is $f = \frac{\sqrt{3}}{2} c_0 \,\mathrm{Hz}$, where c_0 denotes the speed of light in vacuum, and the wave length is $\lambda = 1.1547\,\mathrm{m}$. The angular frequency is $\omega = 2\pi f$ (rad·s^{-1}). The permittivity and the permeability are equal to the constant vacuum values $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ as in [29]. The exact electric and magnetic fields are given as

$$
\begin{aligned}
\mathbf{E}_1(t) &= -\cos(\pi x)\sin(\pi y)\sin(\pi z)\cos(\omega t),\\
\mathbf{E}_2(t) &= 0,\\
\mathbf{E}_3(t) &= \sin(\pi x)\sin(\pi y)\cos(\pi z)\cos(\omega t),\\
\mathbf{H}_1(t) &= -\frac{\pi}{\omega}\sin(\pi x)\cos(\pi y)\cos(\pi z)\sin(\omega t),\\
\mathbf{H}_2(t) &= \frac{2\pi}{\omega}\cos(\pi x)\sin(\pi y)\cos(\pi z)\sin(\omega t),\\
\mathbf{H}_3(t) &= -\frac{\pi}{\omega}\cos(\pi x)\cos(\pi y)\sin(\pi z)\sin(\omega t).
\end{aligned}
$$

For the case of the symplectic time integration method, an upper bound of the largest stable time step (CFL) is given by [84, 48, 129], as

$$
\Delta t \leq \frac{2}{\sqrt{\rho\big(\mathbf{M}_{\varepsilon}^{-1}\mathbf{G}^{\top}\mathbf{M}_{\mu}^{-1}\mathbf{G}\big)}},
$$

where ρ is the spectral radius function. The largest eigenvalue can be efficiently determined by [16] or the power method.

Table 2.1: Absolute Error

Refinement level	Electric and magnetic fields			
	$\Vert\mathbf{E}(t^n) - \mathbf{E}_h^n\Vert_{\mathbf{L}^2(\Omega)}$	$\Vert\mathbf{H}(t^n) - \mathbf{H}_h^n\Vert_{\mathbf{L}^2(\Omega)}$	Δt	Steps
$l = 2$	0.584249	1.9114e-08	0.28230 ns	50
$l = 3$	0.041732	1.1946e-09	0.140619 ns	100
$l = 4$	0.002782	8.2386e-11	0.0706727 ns	250

Example 2.6 Here the backward Euler method is considered. The permittivity and the permeability are constant: $\varepsilon = 2$, $\mu = 1.5$. The initial electric and magnetic fields are obtained by taking the projections (2.26) of the exact

electric and magnetic fields, where the exact fields given by

$$\mathbf{E} = \Big(\sin(2t - 3z), \sin(2t - 3x), \sin(2t - 3y)\Big)^T,$$
$$\mathbf{H} = \Big(\sin(2t - 3y), \sin(2t - 3z), \sin(2t - 3x)\Big)^T,$$
$$\mathbf{J} = \Big(\sin(2t - 3z), \sin(2t - 3x), \sin(2t - 3y)\Big)^T.$$

Theorem 2.1 states that the problem is well-posed. An analogous result for the Rothe method is given in Theorem 2.2. The Theorems 2.3 and 2.4 present a priori estimates of the absolute error. These results show that we get optimal solutions within the selected finite element spaces, whereas many existing methods exhibit spurious solutions e.g. [103, 84, 112, 33, 129, 17, 66, 7, 134, 96, 44, 64, 92, 57, 19] [99, 100, 8, 9, 91] also because they solve other than the direct Maxwell's problem (2.1)-(2.3). We measured the $\mathbf{L}^2$-norm of the error for a sequence of successively refined meshes starting from a uniform coarse mesh. The refinement level at $l = 1$ shows the initial geometry of the mesh, and the levels at $l = 2, l = 3$, and $l = 4$ show the uniform refinement simultaneously at the subsequent 2nd, 3rd and 4th steps. We summarize the obtained absolute errors for the fourth order symplectic integration method in Table 2.1. The Table 2.1 shows that the symplectic method is conditionally stable and its order of the convergence is approximately equal to 4. Fig 2.1 illustrates that the symplectic method conserves the energy. This is an additional aspect to underline the good accuracy of our numerical results.

The snapshot of the electric and magnetic fields depicted in Fig. 2.2 is taken at the final time step $N = 100$, using the time step size $\Delta t = 0.03125$, by employing the backward Euler method for the beam tetrahedron meshes. Fig. 2.3 shows the initial values of the electric and magnetic fields at the first time step by taking the projections (2.26), and by employing the backward Euler method for the Fichera mesh (3D L-shaped domain). The time step size is $\Delta t = 0.0005$ in Fig. 2.3. Different orientations of the Fichera mesh are illustrated in Figs. 2.4. Furthermore, the electric and magnetic fields at time step $N = 50$ are shown in Fig. 2.4, by employing the backward Euler for $\Delta t = 0.0005$. Both the absolute errors and the conservation property of energy are determined and are strictly fulfilled as the electric and magnetic fields visualization in 3D underline for our cases, in contrast to [123]. This is clear from Table 2.1 and the conservation property of energy, see Fig. 2.1. In addition, the electric and magnetic fields are visualized at initial and final time steps for beam and Fichera meshes.

The backward Euler method is unconditionally stable and computationally expensive. In contrast to the backward Euler method the symplectic method is conditionally stable. We conclude that our proposed time domain finite element method methods possess good accuracy and no spurious solutions in 3D complex geometries, and allow to treat the systems of Maxwell's equations directly and more efficiently than many existing methods such as A-formulation, $A - \mathbf{\Phi}$ method, operator-form, electric field formulation, magnetic field formation and decoupled scheme (explicit magnetic field), for details see [103, 84, 112, 33, 129, 17, 66, 7, 134, 96, 44, 64, 92, 57, 19, 99, 100, 8, 9, 91]. Moreover the proposed methods are a good basis for the development of energy conserving methods for nonlinear problems in optics and photonics. The A-formulation scheme [134] also caused spurious solution in the nonlinear case. Moreover, our proposed method could replace the A-formulation [134] to solve the fully non-linear system of High-Power microwave air breakdown without spurious solutions. We have also obtained some first parallel results, see [11].

2.4 Summary

The Chapter 2 summarizes some time domain finite element methods for the system of Maxwell's equations in three dimensions, where the electric and magnetic fields are discretized by means of different finite element spaces. The time domain mixed finite element methods have the advantage of being substantially more powerful and reliable than FDTD or other existing methods with respect to error estimates and numerical experiment, because they directly solve the system for electric and magnetic field intensities. Moreover the Rothe method and fully discrete error estimation yield optimal solutions.

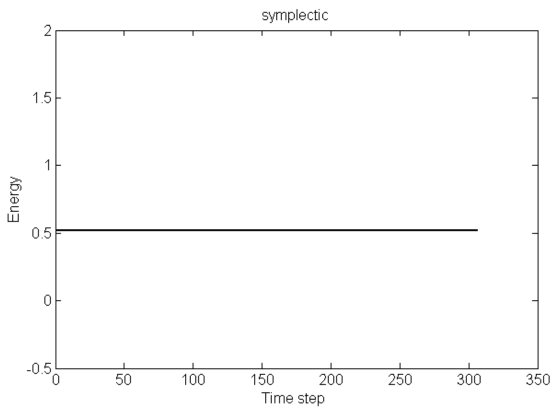
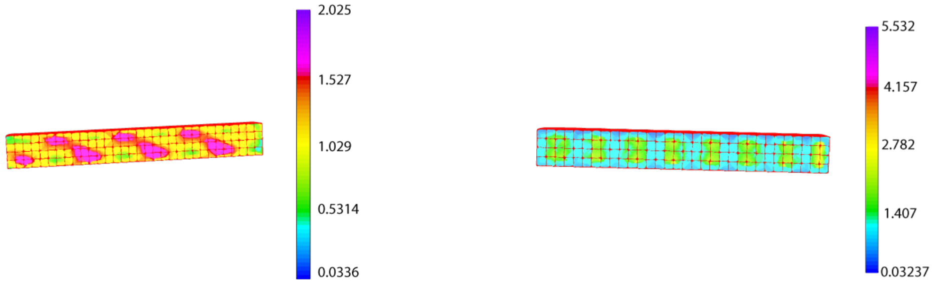
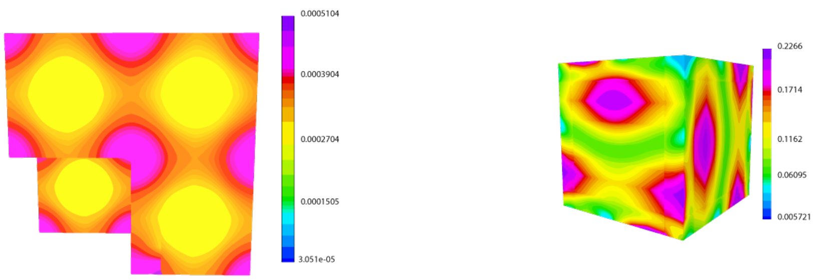


Figure 2.1: The energy of the system remains constant if the symplectic time integration method is applied



(a) Electric Field (b) Magnetic Field

Figure 2.2: The snapshot is taken at the final time step ($N = 100$) for the electric and magnetic fields, by employing the backward Euler method for the beam tetrahedron. The time step size is $\Delta t = 0.03125$

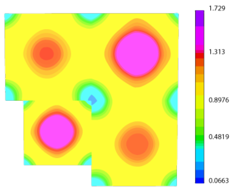


(a) Electric Field (b) Magnetic Field

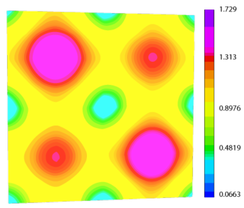
Figure 2.3: The snapshot is taken at the first time step ($n = 1$) for the electric and magnetic fields by taking the projections (2.26), and employing the backward Euler method for the Fichera mesh. The time step size is $\Delta t = 0.0005$

To the best of our knowledge, fully discrete error estimates, simulations and visualizations of the type presented in the Chapter 2, using the Nédeléc curl-conforming and Raviart Thomas div-conforming elements with backward Euler temporal discretization for the system of Maxwell's equations, were not yet available. Numerical examples are given for the fully discrete problems.

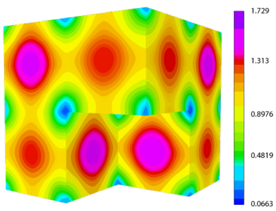
The presented symplectic time integration method is accurate up to fourth order in time, conserves the energy and is conditionally stable. The backward Euler method is unconditionally stable. The computed electric and magnetic fields are visualized at intermediate and final time steps.



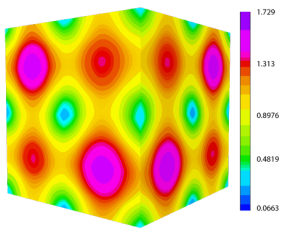
(a) Snapshot of top



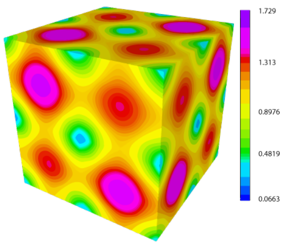
(b) Snapshot of bottom



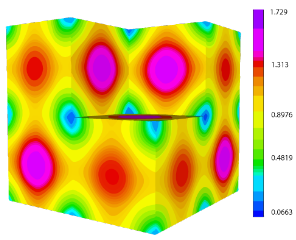
(c) Snapshot of top and front



(d) Snapshot of back and bottom

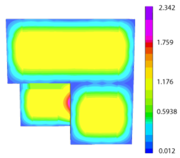


(e) Snapshot of back, left and bottom

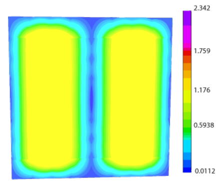


(f) Snapshot of right and front

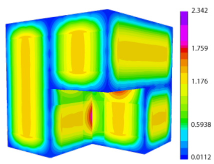
Figure 2.4: The scale shows the values of electric fields at the final time step ($N = 50$), by employing the backward Euler method. The time step size is $\Delta t = 0.0005$



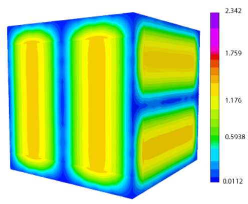
(a) Snapshot of top



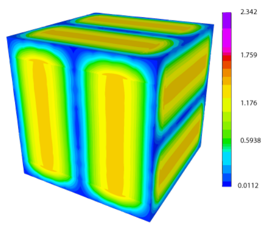
(b) Snapshot of bottom



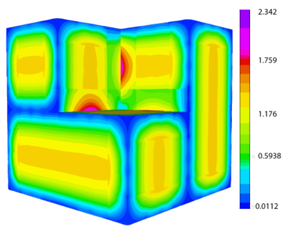
(c) Snapshot of top and front



(d) Snapshot of back and bottom



(e) Snapshot of back, left and bottom



(f) Snapshot of right and front

Figure 2.5: The scale shows the values of magnetic fields at the final time step ($N = 50$), by employing the backward Euler method. The time step size is $\Delta t = 0.0005$

Chapter 3

Nonlinear Maxwell's Equations in Optics and Photonics

Let Ω be a smooth, simply connected domain in $\mathbb{R}^3$ with boundary Γ and unit outward normal $\mathbf{n}$. Let $\mathbf{D} = \mathbf{D}(\mathbf{x}, t)$, $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$, $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H} = \mathbf{H}(\mathbf{x}, t)$ represent the displacement field, magnetic induction, electric and magnetic field intensities respectively, where $\mathbf{x} \in \Omega$ and the time variable t ranges in some interval $(0, T)$, $T > 0$. Given an electric current density $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$, we write the time-dependent Maxwell's equations as

$$\partial_t \mathbf{D} - \nabla \times \mathbf{H} = \mathbf{J} \ \text{ in } \Omega \times (0, T), \tag{3.1}$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 \ \text{ in } \Omega \times (0, T), \tag{3.2}$$

where the following constitutive relations hold:

$$\mathbf{B} := \mu_0 \mathbf{H}, \tag{3.3}$$

$$\mathbf{D} := \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}). \tag{3.4}$$

$\varepsilon_0 > 0$ and $\mu_0 > 0$ are the vacuum permittivity and permeability respectively. Often the nonlinear constitutive relation for the polarization $\mathbf{P} = \mathbf{P}(\mathbf{E})$ is approximated by a truncated Taylor series [26]. In case of an isotropic material, it takes the form,

$$\mathbf{P}(\mathbf{E}) := \varepsilon_0 \Big(\chi^{(1)} \mathbf{E} + \chi^{(3)} \mathbf{E}^3 \Big), \tag{3.5}$$

where, in general, $\chi^{(i)} : \Omega \to (\mathbb{R}^3)^{i+1}$ are the media susceptibility tensors $i = 1, 3$. Here we further restrict the model to more symmetric materials so that the second term in (3.5) takes the form $\chi^{(3)} |\mathbf{E}|^2 \mathbf{E}$ with a nonnegative scalar coefficient $\chi^{(3)} : \Omega \to \mathbb{R}$. We also assume that $\chi^{(1)}$ is a positive scalar coefficient $\chi^{(1)} : \Omega \to \mathbb{R}$. For $\chi^{(3)} = 0$, we obtain the linear Maxwell's equations. Thus the nonlinear Maxwell's equations (3.1)–(3.5) can be rewritten as:

$$\partial_t \mathbf{D} - \nabla \times \mathbf{H} = \mathbf{J} \ \text{ in } \Omega \times (0,T), \tag{3.6}$$

$$\mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} = 0 \ \text{ in } \Omega \times (0,T), \tag{3.7}$$

where

$$\mathbf{D} = \varepsilon_0 \Big((1+\chi^{(1)})\mathbf{E} + \chi^{(3)} |\mathbf{E}|^2 \, \mathbf{E} \Big),$$

and its derivative

$$\partial_t \mathbf{D} = \varepsilon_0 \Big((1+\chi^{(1)})\partial_t \mathbf{E} + \chi^{(3)} \partial_t [|\mathbf{E}|^2 \, \mathbf{E}] \Big). \tag{3.8}$$

A perfect conducting boundary condition on Ω is assumed so that

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{ on } \Gamma \times (0,T).$$

In addition, initial conditions have to be specified so that

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}) \quad \text{and} \quad \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega,$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are given functions on Γ, and $\mathbf{H}_0$ satisfies

$$\nabla \cdot \mu_0 \mathbf{H}_0 = 0 \ \text{ in } \Omega, \ \mathbf{H}_0 \cdot \mathbf{n} = 0 \ \text{ on } \Gamma. \tag{3.9}$$

The divergence-free condition in (3.9) together with (3.7) implies that

$$\nabla \cdot \mu_0 \mathbf{H} = 0 \quad \text{ in } \Omega \times (0,T). \tag{3.10}$$

3.1 A Weak Formulation.

We assume that a unique solution $(\mathbf{E},\mathbf{H}) \in \big(C^1(0,T,\mathbf{L}^2(\Omega)) \cap C^1(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))\big) \times \big(C(0,T,\mathbf{H}(\mathrm{curl}\,;\Omega)) \cap C^1(0,T,\mathbf{L}^2(\Omega))\big)$ of the nonlinear Maxwell's equations (3.6)–(3.8) exists.

We multiply equation (3.6) and (3.8) by a test function $\mathbf{\Psi} \in \mathbf{L}^2(\Omega)$ and integrate over Ω. Similarly we multiply (3.7) by test function $\mathbf{\Phi} \in \mathbf{H}(\mathrm{curl}\,;\Omega)$, integrating the result over Ω and integrate by parts the second term of equation (3.7). This shows that it is a natural to look for a weak solution $(\mathbf{D},\mathbf{E},\mathbf{H}) \in \big(C^1(0,T,\mathbf{L}^2(\Omega)) \times \big(C^1(0,T,\mathbf{L}^2(\Omega)) \cap C^1(0,T,$

$\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))\big) \times \big(C(0,T,\mathbf{H}(\mathrm{curl}\,;\Omega)) \cap C^1(0,T,\mathbf{L}^2(\Omega))\big)$ of (3.6)–(3.8) such that:

$$(\partial_t \mathbf{D}, \mathbf{\Psi}) - (\nabla \times \mathbf{H}, \mathbf{\Psi}) = (\mathbf{J}, \mathbf{\Psi}) \qquad \forall\, \mathbf{\Psi} \in \mathbf{L}^2(\Omega), \tag{3.11}$$

$$(\partial_t \mathbf{D}, \mathbf{\Psi}) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}) + \big(\varepsilon_0 \chi^{(3)} \partial_t\big(|\mathbf{E}|^2\, \mathbf{E}\big), \mathbf{\Psi}\big) \qquad \forall\, \mathbf{\Psi} \in \mathbf{L}^2(\Omega), \tag{3.12}$$

$$(\mu_0 \partial_t \mathbf{H}, \mathbf{\Phi}) + (\mathbf{E}, \nabla \times \mathbf{\Phi}) = 0 \qquad \forall\, \mathbf{\Phi} \in \mathbf{H}(\mathrm{curl}\,;\Omega). \tag{3.13}$$

Alternatively, the test of the functions $\mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}\,;\Omega)$, $\mathbf{\Phi} \in \mathbf{V} = \mathbf{H}(\mathrm{div};\Omega)$ and integration by parts in the equation (3.6) leads to a weak solution $(\mathbf{D}, \mathbf{E}, \mathbf{H}) \in C^1(0,T,\mathbf{H}_0(\mathrm{curl}\,;\Omega) \times \big(C(0,T,\mathbf{H}_0(\mathrm{curl}\,;\Omega)) \cap C^1(0,T, \mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))\big) \times \big(C(0,T,\mathbf{H}(\mathrm{div};\Omega)) \cap C^1(0,T,\mathbf{L}^2(\Omega))\big)$ of (3.6)–(3.8) such that

$$(\partial_t \mathbf{D}, \mathbf{\Psi}) - (\mathbf{H}, \nabla \times \mathbf{\Psi}) = (\mathbf{J}, \mathbf{\Psi}) \qquad \forall\, \mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}\,;\Omega), \tag{3.14}$$

$$(\partial_t \mathbf{D}, \mathbf{\Psi}) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}) + \big(\varepsilon_0 \chi^{(3)} \partial_t\big(|\mathbf{E}|^2\, \mathbf{E}\big), \mathbf{\Psi}\big) \quad \forall\, \mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}\,;\Omega), \tag{3.15}$$

$$(\mu_0 \partial_t \mathbf{H}, \mathbf{\Phi}) + (\nabla \times \mathbf{E}, \mathbf{\Phi}) = 0 \qquad \forall\, \mathbf{\Phi} \in \mathbf{H}(\mathrm{div};\Omega). \tag{3.16}$$

In both cases, initial conditions have to be satisfied,

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}) \quad \text{and} \quad \mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x}) \quad \text{are in } \Omega, \tag{3.17}$$

where $\mathbf{H}_0$ satisfies (3.9).

Remark 3.1 As a consequence of the embedding (as sets)

$$[C_0^\infty(\Omega)]^3 \subset [C^\infty(\Omega) \cap \mathbf{H}_1(\Omega)]^3 \subset [\mathbf{H}_1(\Omega)]^3 \subset \mathbf{H}(\mathrm{div};\Omega) \subset \mathbf{L}^2(\Omega),$$

and of the fact that $C_0^\infty(\Omega)$ is dense in $\mathbf{L}^2(\Omega)$, we see that $\mathbf{H}(\mathrm{div};\Omega)$ is a dense subset of $\mathbf{L}^2(\Omega)$. Therefore the test space in (3.16) can be extended to $\mathbf{L}^2(\Omega)$.

Remark 3.2 In case of μ_0 is not a constant but a highly variable function $\mu = \mu(\mathbf{x})$, it is more convenient to use the magnetic flux density $\mathbf{B} = \mu\mathbf{H}$ instead of $\mathbf{H}$ as a dependent variable [100]. In such a case the formulation

(3.14)–(3.16) is replaced by

$$(\partial_t \mathbf{D}, \mathbf{\Psi}) - (\mu^{-1}\mathbf{B}, \nabla \times \mathbf{\Psi}) = (\mathbf{J}, \mathbf{\Psi}) \qquad \forall\, \mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}\,;\Omega), \tag{3.18}$$

$$(\partial_t \mathbf{D}, \mathbf{\Psi}) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}) + \big(\varepsilon_0 \chi^{(3)} \partial_t\big(|\mathbf{E}|^2\, \mathbf{E}\big), \mathbf{\Psi}\big) \quad \forall\, \mathbf{\Psi} \in \mathbf{H}_0(\mathrm{curl}\,;\Omega), \tag{3.19}$$

$$(\partial_t \mathbf{B}, \mathbf{\Phi}) + (\nabla \times \mathbf{E}, \mathbf{\Phi}) = 0 \qquad \forall\, \mathbf{\Phi} \in \mathbf{L}^2(\Omega). \tag{3.20}$$

3.1.1 The Nonlinear Electromagnetic Energy at the Continuous Level

In this section, the nonlinear electromagnetic energy of the systems (3.11)–(3.13) and (3.14)–(3.16) at any time t is defined by

$$\textbf{Energy} := \|\mathbf{E}(t)\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}^2(t)\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}(t)\|^2_{\mu_0}.$$

In the next, we will prove that the nonlinear electromagnetic energy at any time t is bounded.

Theorem 3.3 *If* $\mathbf{J} \in \mathbf{C}(0,T,\mathbf{L}^2_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}(\Omega))$ *and* $\big(\mathbf{E},\mathbf{H}\big) \in \big(\mathbf{C}(0,T,\mathbf{H}_0(\mathrm{curl}\,;\Omega)) \cap \mathbf{C}^1(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))\big) \times \big(\mathbf{C}(0,T,\mathbf{H}(\mathrm{div};\Omega)) \cap \mathbf{C}^1(0,T,\mathbf{L}^2(\Omega))\big)$ *is the corresponding weak solution of the system (3.14)–(3.16), then the nonlinear electromagnetic energy of the system (3.14)–(3.16) at any time t satisfies*

$$\Big(\|\mathbf{E}(t)\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}^2(t)\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}(t)\|^2_{\mu_0}\Big)^{\frac{1}{2}}$$
$$\le \Big(\|\mathbf{E}_0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}_0^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_0\|^2_{\mu_0}\Big)^{\frac{1}{2}} + \int_0^t \|\mathbf{J}(s)\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\, ds.$$

Remark 3.4 An analogous result can be obtained for the weak formulation (3.11)–(3.13).

Proof: Taking $\mathbf{\Psi} = \mathbf{E}$ in (3.14) and (3.15), then we have

$$(\partial_t \mathbf{D}, \mathbf{E}) - (\mathbf{H}, \nabla \times \mathbf{E}) = (\mathbf{J}, \mathbf{E}), \tag{3.21}$$

$$(\partial_t \mathbf{D}, \mathbf{E}) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{E}) + \big(\varepsilon_0\chi^{(3)}\partial_t\big(|\mathbf{E}|^2\, \mathbf{E}\big), \mathbf{E}\big). \tag{3.22}$$

Taking $\mathbf{\Phi} = \mathbf{H}$ in (3.16), then we have

$$(\mu_0 \partial_t \mathbf{H}, \mathbf{H}) + (\nabla \times \mathbf{E}, \mathbf{H}) = 0. \tag{3.23}$$

Substituting (3.22) into (3.21) and adding the result to (3.23), we obtain

$$(\varepsilon_0(1+\chi^{(1)})\partial_t\mathbf{E},\mathbf{E}) + \big(\varepsilon_0\chi^{(3)}\partial_t\big(|\mathbf{E}|^2\,\mathbf{E}\big),\mathbf{E}\big) + (\mu_0\partial_t\mathbf{H},\mathbf{H}) = (\mathbf{J},\mathbf{E}). \quad (3.24)$$

This could be rewritten as

$$\frac{d}{dt}\Big(\frac{1}{2}\|\mathbf{E}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{4}\|\mathbf{E}^2\|^2_{\varepsilon_0\chi^{(3)}} + \frac{1}{2}\|\mathbf{H}\|^2_{\mu_0}\Big) = (\mathbf{J},\mathbf{E}). \quad (3.25)$$

The right-hand side of the equation (3.25) is estimated by means of the Cauchy-Schwarz inequality (Lemma 5.1(3))

$$\begin{aligned}(\mathbf{J},\mathbf{E}) =&((\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}\mathbf{J},\varepsilon_0(1+\chi^{(1)})\mathbf{E})\\ &\leq \|\mathbf{J}\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\,\|\mathbf{E}\|_{\varepsilon_0(1+\chi^{(1)})}.\end{aligned}$$

Then, we get from equation (3.25)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{E}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}\|^2_{\mu_0}\Big) &\leq |\mathbf{J}\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\Big(\|\mathbf{E}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big)^{\frac{1}{2}}\\ &\leq \|\mathbf{J}\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\Big(\|\mathbf{E}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}\|^2_{\mu_0}\Big)^{\frac{1}{2}}.\end{aligned}$$

This implies

$$\begin{aligned}&\frac{1}{2}\Big(\|\mathbf{E}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}\|^2_{\mu_0}\Big)^{-\frac{1}{2}}\\ &\times\frac{d}{dt}\Big(\|\mathbf{E}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}\|^2_{\mu_0}\Big) \leq \|\mathbf{J}\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}.\end{aligned}$$

By the chain rule, we see that

$$\frac{d}{dt}\Big(\|\mathbf{E}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}\|^2_{\mu_0}\Big)^{\frac{1}{2}} \leq \|\mathbf{J}\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}.$$

Integrating both sides from 0 to t

$$\begin{aligned}&\Big(|\mathbf{E}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}\|^2_{\mu_0}\Big)^{\frac{1}{2}}\\ &-\Big(\|\mathbf{E}_0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}_0^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_0\|^2_{\mu_0}\Big)^{\frac{1}{2}} \leq \int_0^t \|\mathbf{J}(s)\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\,ds,\end{aligned}$$

which complete the proof together with initial conditions. □

3.2 Spatial Discretization

Let, $\mathbf{W}_h \subset \mathbf{L}^2(\Omega)$, $\mathbf{U}_h \subset \mathbf{H}(\mathrm{curl}\,;\Omega)$, $\mathbf{U}_{0h} \subset \mathbf{H}_0(\mathrm{curl}\,;\Omega)$, and $\mathbf{V}_h \subset \mathbf{V}$ be finite dimensional subspaces of the given spaces.
The semi-discrete in space problem for the system (3.11)–(3.13) consists in determining elements $(\mathbf{D}_h, \mathbf{E}_h, \mathbf{H}_h) \in C^1(0,T,\mathbf{W}_h)\times C^1(0,T,\mathbf{W}_h)\times C^1(0,T,\mathbf{U}_h)$ such that

$$(\partial_t \mathbf{D}_h, \mathbf{\Psi}_h) - (\nabla \times \mathbf{H}_h, \mathbf{\Psi}_h) = (\mathbf{J}_h, \mathbf{\Psi}_h) \qquad \forall \mathbf{\Psi}_h \in \mathbf{W}_h, \tag{3.26}$$

$$(\partial_t \mathbf{D}_h, \mathbf{\Psi}) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}_h, \mathbf{\Psi}_h) + \left(\varepsilon_0 \chi^{(3)} \partial_t \left[|\mathbf{E}_h|^2\, \mathbf{E}_h\right], \mathbf{\Psi}_h\right) \quad \forall\, \mathbf{\Psi}_h \in \mathbf{W}_h, \tag{3.27}$$

$$(\mu_0 \partial_t \mathbf{H}_h, \mathbf{\Phi}_h) + (\mathbf{E}_h, \nabla \times \mathbf{\Phi}_h) = 0 \qquad \forall \mathbf{\Phi}_h \in \mathbf{U}_h. \tag{3.28}$$

For the equations (3.14)–(3.16), the semi-discrete problem involves the determination of elements $(\mathbf{D}_h, \mathbf{E}_h, \mathbf{H}_h) \in C^1(0,T,\mathbf{U}_{0h})\times C^1(0,T,\mathbf{U}_{0h})\times C^1(0,T,\mathbf{V}_h)$ satisfying

$$(\partial_t \mathbf{D}_h, \mathbf{\Psi}_h) - (\mathbf{H}_h, \nabla \times \mathbf{\Psi}_h) = (\mathbf{J}_h, \mathbf{\Psi}_h) \qquad \forall \mathbf{\Psi}_h \in \mathbf{U}_{0h}, \tag{3.29}$$

$$(\partial_t \mathbf{D}_h, \mathbf{\Psi}) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}_h, \mathbf{\Psi}_h) + \left(\varepsilon_0 \chi^{(3)} \partial_t \left[|\mathbf{E}_h|^2\, \mathbf{E}_h\right], \mathbf{\Psi}_h\right) \quad \forall\, \mathbf{\Psi}_h \in \mathbf{U}_{0h}, \tag{3.30}$$

$$(\mu_0 \partial_t \mathbf{H}_h, \mathbf{\Phi}_h) + (\nabla \times \mathbf{E}_h, \mathbf{\Phi}_h) = 0 \qquad \forall \mathbf{\Phi}_h \in \mathbf{V}_h. \tag{3.31}$$

In both cases, the initial conditions read formally as

$$\mathbf{E}_h(\mathbf{x},0) = \mathbf{E}_{0h}(\mathbf{x}) \quad \text{ and } \quad \mathbf{H}_h(\mathbf{x},0) = \mathbf{H}_{0h}(\mathbf{x}),$$

where the concrete definitions of the discrete initial data $(\mathbf{E}_{0h}, \mathbf{H}_{0h}) \in \mathbf{W}_h \times \mathbf{U}_h$ or $(\mathbf{E}_{0h}, \mathbf{H}_{0h}) \in \mathbf{U}_{0h} \times \mathbf{V}_h$ will be given later.

3.2.1 The Nonlinear Electromagnetic Energy at the Semi-Discrete Level

The nonlinear electromagnetic energy at the semi-discrete level of the systems (3.26)–(3.28) and (3.29)–(3.31) at time t is defined by

$$\mathbf{Energy}_h := \|\mathbf{E}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|\mathbf{E}_h^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h\|^2_{\mu_0}.$$

In this section, we will show that the nonlinear electromagnetic energy at the semi-discrete level of the systems (3.26)–(3.28) and (3.29)–(3.31) at time t is bounded.

Theorem 3.5 *If* $\mathbf{J}_h \in C^1(0,T,\mathbf{U}_{0h})$ *and* $\big(\mathbf{E}_h,\mathbf{H}_h\big) \in C^1(0,T,\mathbf{U}_{0h})\times C^1(0,T,\mathbf{V}_h)$ *is the corresponding finite element solution at semi-discrete level of the system (3.29)–(3.31), then the nonlinear electromagnetic energy of the system (3.29)–(3.31) at any time t satisfies*

$$\Big(\|\mathbf{E}_h(t)\|^2_{\varepsilon_0(1+\chi^{(1)})}+\frac{3}{2}\|\mathbf{E}_h^2(t)\|^2_{\varepsilon_0\chi^{(3)}}+\|\mathbf{H}_h(t)\|^2_{\mu_0}\Big)^{\frac{1}{2}}$$
$$\leq\Big(\|\mathbf{E}_{0h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\frac{3}{2}\|\mathbf{E}_{0h}^2\|^2_{\varepsilon_0\chi^{(3)}}+\|\mathbf{H}_{0h}\|^2_{\mu_0}\Big)^{\frac{1}{2}}+\int_0^t\|\mathbf{J}_h(s)\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\,ds.$$

Remark 3.6 An analogous result can be obtained for the system of equations (3.26)–(3.28).

Proof: Taking $\mathbf{\Psi}_h=\mathbf{E}_h$ in (3.29) and (3.30), then these can be rewritten as

$$(\partial_t\mathbf{D}_h,\mathbf{E}_h)-(\mathbf{H}_h,\nabla\times\mathbf{E}_h)=(\mathbf{J}_h,\mathbf{E}_h),\tag{3.32}$$
$$(\partial_t\mathbf{D}_h,\mathbf{E}_h)=(\varepsilon_0(1+\chi^{(1)})\partial_t\mathbf{E}_h,\mathbf{E}_h)+(\varepsilon_0\chi^{(3)}\partial_t\big(|\mathbf{E}_h|^2\,\mathbf{E}_h\big),\mathbf{E}_h).\tag{3.33}$$

Substituting equations (3.32) and (3.33), we get

$$(\varepsilon_0(1+\chi^{(1)})\partial_t\mathbf{E}_h,\mathbf{E}_h)+(\varepsilon_0\chi^{(3)}\partial_t\big(|\mathbf{E}_h|^2\,\mathbf{E}_h\big),\mathbf{E}_h)-(\mathbf{H}_h,\nabla\times\mathbf{E}_h)=(\mathbf{J}_h,\mathbf{E}_h).\tag{3.34}$$

Taking $\mathbf{\Phi}_h=\mathbf{H}_h$ in (3.31), it can be written as

$$(\mu_0\partial_t\mathbf{H}_h,\mathbf{H}_h)+(\nabla\times\mathbf{E}_h,\mathbf{H}_h)=0.\tag{3.35}$$

Furthermore, adding (3.34) and (3.35), we get

$$(\varepsilon_0(1+\chi^{(1)})\partial_t\mathbf{E}_h,\mathbf{E}_h)+(\varepsilon_0\chi^{(3)}\partial_t\big(|\mathbf{E}_h|^2\,\mathbf{E}_h\big),\mathbf{E}_h)+(\mu_0\partial_t\mathbf{H}_h,\mathbf{H}_h)=(\mathbf{J}_h,\mathbf{E}_h).$$

This leads to

$$\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{E}_h\|^2_{\varepsilon_0(1+\chi^{(1)})}+\frac{3}{2}\|\mathbf{E}_h^2\|^2_{\varepsilon_0\chi^{(3)}}+\|\mathbf{H}_h\|^2_{\mu_0}\Big)=(\mathbf{J}_h,\mathbf{E}_h).\tag{3.36}$$

The right-hand side of the equation (3.36) is estimated by means of the Cauchy-Schwarz inequality (Lemma 5.1(3))

$$\begin{aligned}(\mathbf{J}_h,\mathbf{E}_h)=&((\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}\mathbf{J}_h,\varepsilon_0(1+\chi^{(1)})\mathbf{E}_h)\\ &\leq\|\mathbf{J}_h\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\,\|\mathbf{E}_h\|_{\varepsilon_0(1+\chi^{(1)})}.\end{aligned}$$

Then we can rewrite the equation (3.36)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{E}_h\|^2_{\varepsilon_0(1+\chi^{(1)})}+\frac{3}{2}\|\mathbf{E}_h^2\|^2_{\varepsilon_0\chi^{(3)}}+\|\mathbf{H}_h\|^2_{\mu_0}\Big)\\
&\qquad\le \|\mathbf{J}_h\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\left(\|\mathbf{E}_h\|^2_{\varepsilon_0(1+\chi^{(1)})}\right)^{\frac{1}{2}}\\
&\qquad\le \|\mathbf{J}_h\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\left(\|\mathbf{E}_h\|^2_{\varepsilon_0(1+\chi^{(1)})}+\frac{3}{2}\|\mathbf{E}_h^2\|^2_{\varepsilon_0\chi^{(3)}}+\|\mathbf{H}_h\|^2_{\mu_0}\right)^{\frac{1}{2}}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\frac{1}{2}\Big(\|\mathbf{E}_h\|^2_{\varepsilon_0(1+\chi^{(1)})}+\frac{3}{2}\|\mathbf{E}_h^2\|^2_{\varepsilon_0\chi^{(3)}}+\|\mathbf{H}_h\|^2_{\mu_0}\Big)^{-\frac{1}{2}}\\
&\cdot\frac{d}{dt}\Big(\|\mathbf{E}_h\|^2_{\varepsilon_0(1+\chi^{(1)})}+\frac{3}{2}\|\mathbf{E}_h^2\|^2_{\varepsilon_0\chi^{(3)}}+\|\mathbf{H}_h\|^2_{\mu_0}\Big)\le\|\mathbf{J}_h\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}.
\end{aligned}
$$

By chain rule, we see that

$$
\frac{d}{dt}\Big(\|\mathbf{E}_h\|^2_{\varepsilon_0(1+\chi^{(1)})}+\frac{3}{2}\|\mathbf{E}_h^2\|^2_{\varepsilon_0\chi^{(3)}}+\|\mathbf{H}_h\|^2_{\mu_0}\Big)^{\frac{1}{2}}\le\|\mathbf{J}_h\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}.
$$

Integrating both sides from 0 to t

$$
\begin{aligned}
&\Big(|\mathbf{E}_h\|^2_{\varepsilon_0(1+\chi^{(1)})}+\frac{3}{2}\|\mathbf{E}_h^2\|^2_{\varepsilon_0\chi^{(3)}}+\|\mathbf{H}_h\|^2_{\mu_0}\Big)^{\frac{1}{2}}\\
&-\Big(\|\mathbf{E}_{0h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+(\frac{1}{2}\|\mathbf{E}_{0h}^2\|^2_{\varepsilon_0\chi^{(3)}}+\|\mathbf{H}_{0h}\|^2_{\mu_0}\Big)^{\frac{1}{2}}\\
&\le\int_0^t\|\mathbf{J}_h(s)\|_{(\varepsilon_0+\varepsilon_0\chi^{(1)})^{-1}}\,ds,
\end{aligned}
$$

which completes the proof. □

In the above theorem, we have demonstrated that the original system on the semi-discrete level maintains its energy stability, either in the implementation of the spatial discretization (3.26)–(3.28) or (3.29)–(3.31).

3.3 Error Estimates for the Semi-Discrete Problem

3.3.1 The Case of the Lee-Madsen Formulation

Theorem 3.7 *Let the weak solution* $\big(\mathbf{E},\mathbf{H}\big) \in \big(\mathbf{C}(0,T,\mathbf{L}^2(\Omega)) \cap \mathbf{C}^1(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))\big) \times \big(\mathbf{C}(0,T,\mathbf{H}(\mathrm{curl}\,;\Omega)) \cap \mathbf{C}^1(0,T,\mathbf{L}^2(\Omega))\big)$ *of the system (3.11)–(3.13), and the finite element solution* $\big(\mathbf{E}_h,\mathbf{H}_h\big) \in C^1(0,T,\mathbf{W}_h) \times C^1(0,T,\mathbf{U}_h)$ *of the system (3.26)–(3.28) respectively exist. Then the following error estimate holds with a constant* $C > 0$ *independent of* h *and* t *such that*

$$\|\mathbf{E}_h - \mathbf{E}\|_{\varepsilon_0} + \|\mathbf{H}_h - \mathbf{H}\|_{\mu_0} \leq Ch^k.$$

Proof: We set $\mathbf{\Psi} = \mathbf{\Psi}_h \in \mathbf{W}_h$ in (3.11)–(3.12)

$$(\partial_t \mathbf{D}, \mathbf{\Psi}_h) - (\nabla \times \mathbf{H}, \mathbf{\Psi}_h) = 0 \qquad \forall\, \mathbf{\Psi}_h \in \mathbf{W}_h,$$
$$(\partial_t \mathbf{D}, \mathbf{\Psi}_h) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}_h) + \big(\varepsilon_0 \chi^{(3)} \partial_t \big[|\mathbf{E}|^2\, \mathbf{E}\big], \mathbf{\Psi}_h\big) \qquad \forall\, \mathbf{\Psi}_h \in \mathbf{W}_h,$$

We set $\mathbf{\Phi} = \mathbf{\Phi}_h \in \mathbf{U}_h$ in (3.13)

$$(\mu_0 \partial_t \mathbf{H}, \mathbf{\Phi}_h) + (\mathbf{E}, \nabla \times \mathbf{\Phi}_h) = 0 \qquad \forall\, \mathbf{\Phi}_h \in \mathbf{U}_h.$$

By means of the projection operators $\tilde{\mathbf{P}}_h$ and Π_{1h} defined in (1.8) and (1.11)–(1.12) respectively, from this we get

$$(\partial_t \mathbf{D}, \mathbf{\Psi}_h) - (\nabla \times \Pi_{1h}\mathbf{H}, \mathbf{\Psi}_h) = (\nabla \times (\mathbf{H} - \Pi_{1h}\mathbf{H}), \mathbf{\Psi}_h) \qquad \forall\, \mathbf{\Psi}_h \in \mathbf{W}_h, \tag{3.37}$$
$$(\partial_t \mathbf{D}, \mathbf{\Psi}_h) = (\varepsilon_0(1+\chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}_h) + \big(\varepsilon_0 \chi^{(3)} \partial_t \big[|\mathbf{E}|^2\, \mathbf{E}\big], \mathbf{\Psi}_h\big) \qquad \forall\, \mathbf{\Psi}_h \in \mathbf{W}_h, \tag{3.38}$$

and

$$\begin{aligned}(\mu_0 \partial_t \Pi_{1h}\mathbf{H}, \mathbf{\Phi}_h) + (\tilde{\mathbf{P}}_h \mathbf{E}, \nabla \times \mathbf{\Phi}_h) &= \mu_0(\Pi_{1h}\partial_t \mathbf{H} - \partial_t \mathbf{H}, \mathbf{\Phi}_h) \\ &+ (\tilde{\mathbf{P}}_h \mathbf{E} - \mathbf{E}, \nabla \times \mathbf{\Phi}_h) \quad \forall\, \mathbf{\Phi}_h \in \mathbf{U}_h.\end{aligned} \tag{3.39}$$

The last term on the right-hand side of (3.37) vanishes thanks to the properties of Π_{1h}, see [102, eq. (2.4)] and (1.11).

The second term on the right-hand side of (3.39) can be omitted because of the commutation property $\partial_t \Pi_{1h}\mathbf{H} = \Pi_{1h}\partial_t \mathbf{H}$. The last term on the right-hand side vanishes thanks to $\nabla \times \mathbf{U}_h \subset \mathbf{W}_h$ and the property (1.8) of $\tilde{\mathbf{P}}_h$.

Therefore the equations (3.37)–(3.39) simplify to

$$(\partial_t \mathbf{D}, \mathbf{\Psi}_h) - (\nabla \times \Pi_{1h}\mathbf{H}, \mathbf{\Psi}_h) = (\nabla \times (\mathbf{H} - \Pi_{1h}\mathbf{H}), \mathbf{\Psi}_h) \qquad \forall\, \mathbf{\Psi}_h \in \mathbf{W}_h, \tag{3.40}$$

$$(\partial_t \mathbf{D}, \mathbf{\Psi}_h) = (\varepsilon_0(1 + \chi^{(1)})\partial_t \mathbf{E}, \mathbf{\Psi}_h) + \big(\varepsilon_0 \chi^{(3)} \partial_t \big[|\mathbf{E}|^2\, \mathbf{E}\big], \mathbf{\Psi}_h\big) \quad \forall\, \mathbf{\Psi}_h \in \mathbf{W}_h, \tag{3.41}$$

$$(\mu_0 \partial_t \Pi_{1h}\mathbf{H}, \mathbf{\Phi}_h) + (\tilde{\mathbf{P}}_h \mathbf{E}, \nabla \times \mathbf{\Phi}_h) = \mu_0(\Pi_{1h}\partial_t \mathbf{H} - \partial_t \mathbf{H}, \mathbf{\Phi}_h) \quad \forall\, \mathbf{\Phi}_h \in \mathbf{U}_h. \tag{3.42}$$

Now, subtracting (3.40)–(3.42) from the system (3.26)–(3.28) and taking into consideration that μ_0 is constant, we obtain:

$$\begin{aligned}
&(\varepsilon_0(1 + \chi^{(1)})(\partial_t \mathbf{E}_h - \partial_t \mathbf{E}), \mathbf{\Psi}_h) + \big(\varepsilon_0 \chi^{(3)} \partial_t \big[|\mathbf{E}_h|^2\, \mathbf{E}_h - |\mathbf{E}|^2\, \mathbf{E}\big], \mathbf{\Psi}_h\big) \\
&- (\nabla \times (\mathbf{H}_h - \Pi_{1h}\mathbf{H}), \mathbf{\Psi}_h) = 0, &(3.43)\\
&\mu_0(\partial_t(\mathbf{H}_h - \Pi_{1h}\mathbf{H}), \mathbf{\Phi}_h) + (\mathbf{E}_h - \tilde{\mathbf{P}}_h \mathbf{E}, \nabla \times \mathbf{\Phi}_h) = \mu_0(\partial_t \mathbf{H} - \Pi_{1h}\partial_t \mathbf{H}, \mathbf{\Phi}_h). &(3.44)
\end{aligned}$$

Now we will deal with the first two terms of (3.43), where we have in mind the choice $\mathbf{\Psi}_h = \mathbf{E}_h - \tilde{\mathbf{P}}_h \mathbf{E}$:

$$\begin{aligned}
&\varepsilon_0(1 + \chi^{(1)})(\partial_t \mathbf{E}_h - \partial_t \mathbf{E}), \mathbf{\Psi}_h + \varepsilon_0 \chi^{(3)} \partial_t \big[|\mathbf{E}_h|^2\, \mathbf{E}_h - |\mathbf{E}|^2\, \mathbf{E}\big] \\
&= \varepsilon_0(1 + \chi^{(1)})(\partial_t \mathbf{E}_h - \partial_t \mathbf{E}), \mathbf{\Psi}_h + \varepsilon_0 \chi^{(3)} \big[|\mathbf{E}_h|^2\, \partial_t \mathbf{E}_h - |\mathbf{E}|^2\, \partial_t \mathbf{E}\big] \\
&+ \varepsilon_0 \chi^{(3)} \big[2\big(\mathbf{E}_h \cdot \partial_t \mathbf{E}_h\big)\, \mathbf{E}_h - 2\big(\mathbf{E} \cdot \partial_t \mathbf{E}\big)\, \mathbf{E}\big] \\
&= \varepsilon_0(1 + \chi^{(1)})(\partial_t \mathbf{E}_h - \partial_t \mathbf{E}), \mathbf{\Psi}_h + \varepsilon_0 \chi^{(3)} \big[|\mathbf{E}_h|^2\, \partial_t \mathbf{E}_h - |\mathbf{E}|^2\, \partial_t \mathbf{E}\big] \\
&+ 2\, \varepsilon_0 \chi^{(3)} \big[\big(\mathbf{E}_h \mathbf{E}_h^T\big)\, \partial_t \mathbf{E}_h - \big(\mathbf{E}\mathbf{E}^T\big)\, \partial_t \mathbf{E}\big] \\
&= \varepsilon_0(1 + \chi^{(1)})\partial_t(\mathbf{E}_h - \mathbf{E}) + \varepsilon_0\, \chi^{(3)}\, \big[|\mathbf{E}_h|^2 \partial_t \mathbf{E}_h - |\mathbf{E}|^2 \partial_t \mathbf{E}\big] \\
&\quad + 2\, \varepsilon_0 \chi^{(3)}\, \big[\mathbf{E}_h \mathbf{E}_h^\top \partial_t \mathbf{E}_h - \mathbf{E}\mathbf{E}^\top \partial_t \mathbf{E}\big] =: \varepsilon_0\, [\delta_1 + \delta_2 + \delta_3].
\end{aligned}$$

The treatment of δ_1 is quite obvious. With $\mathbf{E}_h - \mathbf{E} = \mathbf{\Psi}_h + \tilde{\mathbf{P}}_h \mathbf{E} - \mathbf{E}$ we get

$$\delta_1 = (1 + \chi^{(1)})\partial_t \mathbf{\Psi}_h + (1 + \chi^{(1)})\partial_t(\tilde{\mathbf{P}}_h \mathbf{E} - \mathbf{E}) =: \delta_{11} + \delta_{12}.$$

The term δ_2 is decomposed as follows:

$$\begin{aligned}
\delta_2 &= \chi^{(3)}\, \big[|\mathbf{E}_h|^2 \partial_t \mathbf{E}_h - |\mathbf{E}|^2 \partial_t \mathbf{E}\big] \\
&= \chi^{(3)}\, \big[|\mathbf{E}_h|^2 - |\mathbf{E}|^2\big]\, \partial_t \mathbf{E}_h + \chi^{(3)} |\mathbf{E}|^2 \partial_t(\mathbf{E}_h - \mathbf{E}) \\
&= \chi^{(3)}(\mathbf{E}_h + \mathbf{E})^\top(\mathbf{E}_h - \mathbf{E})\partial_t \mathbf{E}_h + \chi^{(3)} |\mathbf{E}|^2 \partial_t \mathbf{\Psi}_h + \chi^{(3)} |\mathbf{E}|^2 \partial_t(\tilde{\mathbf{P}}_h \mathbf{E} - \mathbf{E}) \\
&=: \delta_{21} + \delta_{22} + \delta_{23}.
\end{aligned}$$

For δ_3, we use the following decomposition:

$$\begin{aligned}
\delta_3 &= 2\chi^{(3)} \left[\mathbf{E}_h\mathbf{E}_h^\top \partial_t \mathbf{E}_h - \mathbf{E}\mathbf{E}^\top \partial_t \mathbf{E}\right] \\
&= 2\chi^{(3)} \left[\mathbf{E}_h\mathbf{E}_h^\top - \mathbf{E}\mathbf{E}^\top\right] \partial_t \mathbf{E}_h + 2\chi^{(3)}\mathbf{E}\mathbf{E}^\top \partial_t(\mathbf{E}_h - \mathbf{E}) \\
&= 2\chi^{(3)}(\mathbf{E}_h - \mathbf{E})\mathbf{E}_h^\top \partial_t \mathbf{E}_h + 2\chi^{(3)}\mathbf{E}(\mathbf{E}_h - \mathbf{E})^\top \partial_t \mathbf{E}_h \\
&+ 2\chi^{(3)}\mathbf{E}\mathbf{E}^\top \partial_t \mathbf{\Psi}_h + 2\chi^{(3)}\mathbf{E}\mathbf{E}^\top \partial_t(\tilde{\mathbf{P}}_h \mathbf{E} - \mathbf{E}) \\
&=: \delta_{31} + \delta_{32} + \delta_{33} + \delta_{34}.
\end{aligned}$$

With these decompositions, equation (3.43) takes the form

$$\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})(\partial_t \mathbf{E}_h - \partial_t \mathbf{E}), \mathbf{\Psi}_h) + \left(\varepsilon_0 \chi^{(3)} \partial_t \left[|\mathbf{E}_h|^2 \mathbf{E}_h - |\mathbf{E}|^2 \mathbf{E}\right], \mathbf{\Psi}_h\right) \\
&- (\nabla \times (\mathbf{H}_h - \Pi_{1h}\mathbf{H}), \mathbf{\Psi}_h) \\
&= \varepsilon_0 \int_\Omega [\delta_{11} + \delta_{22} + \delta_{33}]^\top \mathbf{\Psi}_h d\mathbf{x} \\
&+ \varepsilon_0 \int_\Omega [\delta_{12} + \delta_{21} + \delta_{23} + \delta_{31} + \delta_{32} + \delta_{34}]^\top \mathbf{\Psi}_h d\mathbf{x} \\
&- (\nabla \times (\mathbf{H}_h - \Pi_{1h}\mathbf{H}), \mathbf{\Psi}_h) = 0,
\end{aligned}$$

or, after some rearrangement,

$$\begin{aligned}
&\varepsilon_0 \int_\Omega [\delta_{11} + \delta_{22} + \delta_{33}]^\top \mathbf{\Psi}_h d\mathbf{x} - (\nabla \times (\mathbf{H}_h - \Pi_{1h}\mathbf{H}), \mathbf{\Psi}_h) \\
&= -\varepsilon_0 \int_\Omega [\delta_{12} + \delta_{21} + \delta_{23} + \delta_{31} + \delta_{32} + \delta_{34}]^\top \mathbf{\Psi}_h d\mathbf{x}.
\end{aligned} \tag{3.45}$$

Then:

$$\begin{aligned}
&\varepsilon_0 \int_\Omega [\delta_{11} + \delta_{22} + \delta_{33}]^\top \mathbf{\Psi}_h d\mathbf{x} \\
&= \varepsilon_0 \int_\Omega \left[(1+\chi^{(1)})\partial_t \mathbf{\Psi}_h^\top \mathbf{\Psi}_h + \chi^{(3)}|\mathbf{E}|^2 \partial_t \mathbf{\Psi}_h^\top \mathbf{\Psi}_h + 2\chi^{(3)} \left(\mathbf{E}\mathbf{E}^\top \partial_t \mathbf{\Psi}_h\right)^\top \mathbf{\Psi}_h\right] d\mathbf{x} \\
&= \frac{\varepsilon_0}{2} \int_\Omega \left[(1+\chi^{(1)})\partial_t |\mathbf{\Psi}_h|^2 + \chi^{(3)}|\mathbf{E}|^2 \partial_t |\mathbf{\Psi}_h|^2 + 4\chi^{(3)}\mathbf{E}^\top \partial_t \mathbf{\Psi}_h \mathbf{E}^\top \mathbf{\Psi}_h\right] d\mathbf{x}\,.
\end{aligned}$$

Since

$|\mathbf{E}|^2\partial_t|\mathbf{\Psi}_h|^2 = \partial_t(|\mathbf{E}|^2|\mathbf{\Psi}_h|^2) - \partial_t(|\mathbf{E}|^2)|\mathbf{\Psi}_h|^2$ and $\mathbf{E}^\top \partial_t \mathbf{\Psi}_h = \partial_t(\mathbf{E}^\top \mathbf{\Psi}_h) - \partial_t \mathbf{E}^\top \mathbf{\Psi}_h$, it follows that

$$
\begin{aligned}
&\varepsilon_0 \int_\Omega \left[\delta_{11} + \delta_{22} + \delta_{33}\right]^\top \boldsymbol{\Psi}_h d\mathbf{x} \\
&= \frac{\varepsilon_0}{2} \int_\Omega \Big[(1+\chi^{(1)})\partial_t |\boldsymbol{\Psi}_h|^2 + \chi^{(3)}\partial_t(|\mathbf{E}|^2|\boldsymbol{\Psi}_h|^2) - \chi^{(3)}\partial_t(|\mathbf{E}|^2)|\boldsymbol{\Psi}_h|^2 \\
&+ 4\chi^{(3)}\partial_t(\mathbf{E}^\top\boldsymbol{\Psi}_h)\mathbf{E}^\top\boldsymbol{\Psi}_h - 4\chi^{(3)}\partial_t\mathbf{E}^\top\boldsymbol{\Psi}_h\mathbf{E}^\top\boldsymbol{\Psi}_h\Big]\, d\mathbf{x} \\
&= \frac{\varepsilon_0}{2} \int_\Omega \Big[(1+\chi^{(1)})\partial_t |\boldsymbol{\Psi}_h|^2 + \chi^{(3)}\partial_t(|\mathbf{E}|^2|\boldsymbol{\Psi}_h|^2) - \chi^{(3)}\partial_t(|\mathbf{E}|^2)|\boldsymbol{\Psi}_h|^2 \\
&+ 2\chi^{(3)}\partial_t|\mathbf{E}^\top\boldsymbol{\Psi}_h|^2 - 4\chi^{(3)}\partial_t\mathbf{E}^\top\boldsymbol{\Psi}_h\mathbf{E}^\top\boldsymbol{\Psi}_h\Big]\, d\mathbf{x} \\
&= \frac{\varepsilon_0}{2} \int_\Omega \Big[(1+\chi^{(1)})\partial_t |\boldsymbol{\Psi}_h|^2 + \chi^{(3)}\partial_t(|\mathbf{E}|^2|\boldsymbol{\Psi}_h|^2) + 2\chi^{(3)}\partial_t|\mathbf{E}^\top\boldsymbol{\Psi}_h|^2\Big]\, d\mathbf{x} \\
&- \frac{\varepsilon_0}{2} \int_\Omega \chi^{(3)}\partial_t(|\mathbf{E}|^2)|\boldsymbol{\Psi}_h|^2 d\mathbf{x} - 2\varepsilon_0 \int_\Omega \chi^{(3)}\partial_t\mathbf{E}^\top\boldsymbol{\Psi}_h\mathbf{E}^\top\boldsymbol{\Psi}_h d\mathbf{x}\,.
\end{aligned}
$$

From the estimates

$$
\begin{aligned}
&\left|\frac{\varepsilon_0}{2} \int_\Omega \chi^{(3)}\partial_t(|\mathbf{E}|^2)|\boldsymbol{\Psi}_h|^2 d\mathbf{x}\right| \\
&= \left|\varepsilon_0 \int_\Omega \chi^{(3)}\partial_t\mathbf{E}^\top\mathbf{E}|\boldsymbol{\Psi}_h|^2 d\mathbf{x}\right| \\
&\leq \|\chi^{(3)}\|_{L^\infty(\Omega)} \|\partial_t\mathbf{E}\|_{\mathbf{C}(0,T,L^\infty(\Omega))} \|\mathbf{E}\|_{\mathbf{C}(0,T,L^\infty(\Omega))} \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0} \\
&\leq \|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}\|^2_{\mathbf{C}^1(0,T,L^\infty(\Omega))} \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0}
\end{aligned}
$$

and, analogously,

$$
\left|\varepsilon_0 \int_\Omega \chi^{(3)}\partial_t\mathbf{E}^\top\boldsymbol{\Psi}_h\mathbf{E}^\top\boldsymbol{\Psi}_h d\mathbf{x}\right| \leq \|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}\|^2_{\mathbf{C}^1(0,T,L^\infty(\Omega))} \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0}
$$

we conclude that

$$
\begin{aligned}
&\varepsilon_0 \int_\Omega \left[\delta_{11} + \delta_{22} + \delta_{33}\right]^\top \boldsymbol{\Psi}_h d\mathbf{x} \\
&\geq \frac{\varepsilon_0}{2} \int_\Omega \Big[(1+\chi^{(1)})\partial_t |\boldsymbol{\Psi}_h|^2 + \chi^{(3)}\partial_t(|\mathbf{E}|^2|\boldsymbol{\Psi}_h|^2) + 2\chi^{(3)}\partial_t|\mathbf{E}^\top\boldsymbol{\Psi}_h|^2\Big]\, d\mathbf{x} \\
&- 2\|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}\|^2_{\mathbf{C}^1(0,T,L^\infty(\Omega))} \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0} \\
&= \frac{1}{2}\partial_t \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{\varepsilon_0}{2}\partial_t \int_\Omega \chi^{(3)} \left[|\mathbf{E}|^2|\boldsymbol{\Psi}_h|^2 + 2|\mathbf{E}^\top\boldsymbol{\Psi}_h|^2\right] d\mathbf{x} \\
&- 2\|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}\|^2_{\mathbf{C}^1(0,T,L^\infty(\Omega))} \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0}. \qquad (3.46)
\end{aligned}
$$

For the right-hand side, we have:

$$
\begin{aligned}
& - \varepsilon_0 \int_\Omega \left[\delta_{12} + \delta_{21} + \delta_{23} + \delta_{31} + \delta_{32} + \delta_{34}\right]^\top \mathbf{\Psi}_h d\mathbf{x} \\
&= -\varepsilon_0 \int_\Omega \Big[(1 + \chi^{(1)})\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})^\top \mathbf{\Psi}_h \\
&+ \chi^{(3)}(\mathbf{E}_h + \mathbf{E})^\top(\mathbf{E}_h - \mathbf{E})\partial_t\mathbf{E}_h^\top\mathbf{\Psi}_h + \chi^{(3)}|\mathbf{E}|^2\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})^\top\mathbf{\Psi}_h \\
&+ 2\chi^{(3)}\left((\mathbf{E}_h - \mathbf{E})\mathbf{E}_h^\top\partial_t\mathbf{E}_h\right)^\top \mathbf{\Psi}_h + 2\chi^{(3)}\left(\mathbf{E}(\mathbf{E}_h - \mathbf{E})^\top\partial_t\mathbf{E}_h\right)^\top \mathbf{\Psi}_h \\
&+ 2\chi^{(3)}\Big(\mathbf{E}\mathbf{E}^\top\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})\Big)^\top \mathbf{\Psi}_h\Big] d\mathbf{x} \\
&\leq \varepsilon_0 \int_\Omega \Big[(1 + \chi^{(1)})|\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})||\mathbf{\Psi}_h| \\
&+ \chi^{(3)}|\mathbf{E}_h + \mathbf{E}||\mathbf{E}_h - \mathbf{E}||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h| + \chi^{(3)}|\mathbf{E}|^2|\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})||\mathbf{\Psi}_h| \\
&+ 2\chi^{(3)}\mathbf{E}_h^\top\partial_t\mathbf{E}_h(\mathbf{E}_h - \mathbf{E})^\top\mathbf{\Psi}_h + 2\chi^{(3)}(\mathbf{E}_h - \mathbf{E})^\top\partial_t\mathbf{E}_h\mathbf{E}^\top\mathbf{\Psi}_h \\
&+ 2\chi^{(3)}\mathbf{E}^\top\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})\mathbf{E}^\top\mathbf{\Psi}_h\Big] d\mathbf{x} \\
&\leq \varepsilon_0 \int_\Omega \Big[(1 + \chi^{(1)} + \chi^{(3)}|\mathbf{E}|^2)|\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})||\mathbf{\Psi}_h| \\
&+ \chi^{(3)}|\mathbf{E}_h + \mathbf{E}||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h|^2 + \chi^{(3)}|\mathbf{E}_h + \mathbf{E}||\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h| \\
&+ 2\chi^{(3)}\mathbf{E}_h^\top\partial_t\mathbf{E}_h|\mathbf{\Psi}_h|^2 + 2\chi^{(3)}\mathbf{E}_h^\top\partial_t\mathbf{E}_h(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})^\top\mathbf{\Psi}_h \\
&+ 2\chi^{(3)}\mathbf{\Psi}_h^\top\partial_t\mathbf{E}_h\mathbf{E}^\top\mathbf{\Psi}_h + 2\chi^{(3)}(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})^\top\partial_t\mathbf{E}_h\mathbf{E}^\top\mathbf{\Psi}_h \\
&+ 2\chi^{(3)}\mathbf{E}^\top\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})\mathbf{E}^\top\mathbf{\Psi}_h\Big] d\mathbf{x} \\
&\leq \varepsilon_0 \int_\Omega \Big[(1 + \chi^{(1)} + \chi^{(3)}|\mathbf{E}|^2)|\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})||\mathbf{\Psi}_h| \\
&+ \chi^{(3)}|\mathbf{E}_h||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h|^2 + \chi^{(3)}|\mathbf{E}||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h|^2 \\
&+ \chi^{(3)}|\mathbf{E}_h||\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h| + \chi^{(3)}|\mathbf{E}||\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h| \\
&+ 2\chi^{(3)}|\mathbf{E}_h||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h|^2 + 2\chi^{(3)}|\mathbf{E}_h||\partial_t\mathbf{E}_h||\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}||\mathbf{\Psi}_h| \\
&+ 2\chi^{(3)}|\mathbf{E}||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h|^2 + 2\chi^{(3)}|\mathbf{E}||\partial_t\mathbf{E}_h||\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}||\mathbf{\Psi}_h| \\
&+ 2\chi^{(3)}|\mathbf{E}|^2|\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})||\mathbf{\Psi}_h|\Big] d\mathbf{x} \\
&= \varepsilon_0 \int_\Omega \Big[(1 + \chi^{(1)} + 3\chi^{(3)}|\mathbf{E}|^2)|\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})||\mathbf{\Psi}_h| \\
&+ 3\chi^{(3)}|\mathbf{E}_h||\partial_t\mathbf{E}_h||\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}||\mathbf{\Psi}_h| + 3\chi^{(3)}|\mathbf{E}||\partial_t\mathbf{E}_h||\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}||\mathbf{\Psi}_h| \\
&+ 3\chi^{(3)}|\mathbf{E}_h||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h|^2 + 3\chi^{(3)}|\mathbf{E}||\partial_t\mathbf{E}_h||\mathbf{\Psi}_h|^2\Big] d\mathbf{x}
\end{aligned}
$$

$$
\begin{aligned}
&= \varepsilon_0 \int_\Omega \Big[(1+\chi^{(1)} + 3\chi^{(3)}|\mathbf{E}|^2)|\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})||\boldsymbol{\Psi}_h| \\
&+ 3\chi^{(3)}(|\mathbf{E}_h| + |\mathbf{E}|)|\partial_t \mathbf{E}_h||\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}||\boldsymbol{\Psi}_h| \\
&+ 3\chi^{(3)}(|\mathbf{E}_h| + |\mathbf{E}|)|\partial_t \mathbf{E}_h||\boldsymbol{\Psi}_h|^2 \Big]\, d\mathbf{x} \\
&\leq \left[\|1+\chi^{(1)}\|_{L^\infty(\Omega)} + 3\|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}\|^2_{\mathbf{C}(0,T,L^\infty(\Omega))} \right] \|\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})\|_{\varepsilon_0} \|\boldsymbol{\Psi}_h\|_{\varepsilon_0} \\
&+ 3\|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}_h\|_{\mathbf{C}(0,T,L^\infty(\Omega))} + \|\mathbf{E}\|_{\mathbf{C}(0,T,L^\infty(\Omega))} \right] \|\partial_t \mathbf{E}_h\|_{\mathbf{C}(0,T,L^\infty(\Omega))} \\
&\cdot \|\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}\|_{\varepsilon_0} \|\boldsymbol{\Psi}_h\|_{\varepsilon_0} + 3\|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}_h\|_{\mathbf{C}(0,T,L^\infty(\Omega))} + \|\mathbf{E}\|_{\mathbf{C}(0,T,L^\infty(\Omega))} \right] \\
&\cdot \|\partial_t \mathbf{E}_h\|_{\mathbf{C}(0,T,L^\infty(\Omega))} \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0} \\
&=: C_1 \|\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})\|_{\varepsilon_0} \|\boldsymbol{\Psi}_h\|_{\varepsilon_0} + C_2 \|\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}\|_{\varepsilon_0} \|\boldsymbol{\Psi}_h\|_{\varepsilon_0} + C_3 \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0},
\end{aligned} \tag{3.47}
$$

where the positive constants C_1, C_2, C_3 depend on certain norms of $\chi^{(1)}$, $\chi^{(3)}$, $\mathbf{E}$, and $\mathbf{E}_h$. Combining the estimates (3.46) and (3.47) with (3.45), we get

$$
\begin{aligned}
&\frac{1}{2}\partial_t \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{\varepsilon_0}{2}\partial_t \int_\Omega \chi^{(3)} \left[|\mathbf{E}|^2 |\boldsymbol{\Psi}_h|^2 + 2|\mathbf{E}^\top \boldsymbol{\Psi}_h|^2 \right] d\mathbf{x} \\
&- 2\|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}\|^2_{\mathbf{C}^1(0,T,L^\infty(\Omega))} \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0} - (\nabla \times (\mathbf{H}_h - \Pi_{1h}\mathbf{H}), \boldsymbol{\Psi}_h) \\
&\leq \varepsilon_0 \int_\Omega \left[\delta_{11} + \delta_{22} + \delta_{33} \right]^\top \boldsymbol{\Psi}_h d\mathbf{x} - (\nabla \times (\mathbf{H}_h - \Pi_{1h}\mathbf{H}), \boldsymbol{\Psi}_h) \\
&= -\varepsilon_0 \int_\Omega \left[\delta_{12} + \delta_{21} + \delta_{23} + \delta_{31} + \delta_{32} + \delta_{34} \right]^\top \boldsymbol{\Psi}_h d\mathbf{x} \\
&\leq C_1 \|\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})\|_{\varepsilon_0} \|\boldsymbol{\Psi}_h\|_{\varepsilon_0} + C_2 \|\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}\|_{\varepsilon_0} \|\boldsymbol{\Psi}_h\|_{\varepsilon_0} + C_3 \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0}.
\end{aligned}
$$

This finally leads to

$$
\begin{aligned}
&\frac{1}{2}\partial_t \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{\varepsilon_0}{2}\partial_t \int_\Omega \chi^{(3)} \left[|\mathbf{E}|^2 |\boldsymbol{\Psi}_h|^2 + 2|\mathbf{E}^\top \boldsymbol{\Psi}_h|^2 \right] d\mathbf{x} \\
&- (\nabla \times (\mathbf{H}_h - \Pi_{1h}\mathbf{H}), \boldsymbol{\Psi}_h) \\
&\leq \Big[C_1 \|\partial_t(\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E})\|_{\varepsilon_0} + C_2 \|\tilde{\mathbf{P}}_h\mathbf{E} - \mathbf{E}\|_{\varepsilon_0} \Big] \|\boldsymbol{\Psi}_h\|_{\varepsilon_0} + C_4 \|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0},
\end{aligned}
$$

where

$$
C_4 := C_3 + 2\|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}\|^2_{\mathbf{C}^1(0,T,L^\infty(\Omega))}.
$$

Now we consider (3.44) with $\boldsymbol{\Phi}_h = \mathbf{H}_h - \Pi_{1h}\mathbf{H}$ and get

$$
\begin{aligned}
\frac{1}{2}\partial_t \|\boldsymbol{\Phi}_h\|^2_{\mu_0} + (\mathbf{E}_h - \tilde{\mathbf{P}}_h\mathbf{E}, \nabla \times \boldsymbol{\Phi}_h) &= \mu_0 (\partial_t \mathbf{H} - \Pi_{1h}\partial_t \mathbf{H}, \boldsymbol{\Phi}_h) \\
&\leq \|\partial_t \mathbf{H} - \Pi_{1h}\partial_t \mathbf{H}\|_{\mu_0} \|\boldsymbol{\Phi}_h\|_{\mu_0}.
\end{aligned}
$$

Adding both inequalities and making use of the commutation property of $\tilde{\mathbf{P}}_h$, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{1}{2}\partial_t\|\boldsymbol{\Phi}_h\|^2_{\mu_0} + \frac{\varepsilon_0}{2}\partial_t\int_\Omega \chi^{(3)}\left[|\mathbf{E}|^2|\boldsymbol{\Psi}_h|^2 + 2|\mathbf{E}^\top\boldsymbol{\Psi}_h|^2\right] d\mathbf{x}\\
&\le \left[C_1\|\partial_t\mathbf{E} - \tilde{\mathbf{P}}_h\partial_t\mathbf{E}\|_{\varepsilon_0} + C_2\|\mathbf{E} - \tilde{\mathbf{P}}_h\mathbf{E}\|_{\varepsilon_0}\right]\|\boldsymbol{\Psi}_h\|_{\varepsilon_0}\\
&+ \|\partial_t\mathbf{H} - \Pi_{1h}\partial_t\mathbf{H}\|_{\mu_0}\|\boldsymbol{\Phi}_h\|_{\mu_0} + C_4\|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0}.
\end{aligned}
$$

The projection errors can be estimated by means of (1.10) and (1.13), that is, for $\mathbf{E}, \partial_t\mathbf{E} \in \mathbf{H}^k(\Omega)$ and $\partial_t\mathbf{H} \in \mathbf{H}^{k+1}(\Omega)$, we have that

$$
\begin{aligned}
\|\mathbf{E} - \tilde{\mathbf{P}}_h\mathbf{E}\|_{\varepsilon_0} &\le C\sqrt{\varepsilon_0}\, h^k\|\mathbf{E}\|_{\mathbf{H}^k(\Omega)} \le C\sqrt{\varepsilon_0}\, h^k\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{H}^k(\Omega))},\\
\|\partial_t\mathbf{E} - \tilde{\mathbf{P}}_h\partial_t\mathbf{E}\|_{\varepsilon_0} &\le C\sqrt{\varepsilon_0}\, h^k\|\partial_t\mathbf{E}\|_{\mathbf{H}^k(\Omega)} \le C\sqrt{\varepsilon_0}\, h^k\|\partial_t\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{H}^k(\Omega))},
\end{aligned}
\tag{3.48}
$$

$$
\|\partial_t\mathbf{H} - \Pi_{1h}\partial_t\mathbf{H}\|_{\mu_0} \le C\sqrt{\mu_0}\, h^k\|\partial_t\mathbf{H}\|_{\mathbf{H}^{k+1}(\Omega)} \le C\sqrt{\mu_0}\, h^k\|\partial_t\mathbf{H}\|_{\mathbf{C}(0,T,\mathbf{H}^{k+1}(\Omega))}.
\tag{3.49}
$$

In this way the above estimate can be written as

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{1}{2}\partial_t\|\boldsymbol{\Phi}_h\|^2_{\mu_0} + \frac{\varepsilon_0}{2}\partial_t\int_\Omega \chi^{(3)}\left[|\mathbf{E}|^2|\boldsymbol{\Psi}_h|^2 + 2|\mathbf{E}^\top\boldsymbol{\Psi}_h|^2\right] d\mathbf{x}\\
&\le C_5 h^k\left[\|\boldsymbol{\Psi}_h\|_{\varepsilon_0} + \|\boldsymbol{\Phi}_h\|_{\mu_0}\right] + C_4\|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0}.
\end{aligned}
$$

Setting

$$
w_h(t) := \sqrt{\|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0} + \|\boldsymbol{\Phi}_h\|^2_{\mu_0}},
$$

we get

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{1}{2}\partial_t\|\boldsymbol{\Phi}_h\|^2_{\mu_0} + \frac{\varepsilon_0}{2}\partial_t\int_\Omega \chi^{(3)}\left[|\mathbf{E}|^2|\boldsymbol{\Psi}_h|^2 + 2|\mathbf{E}^\top\boldsymbol{\Psi}_h|^2\right] d\mathbf{x}\\
&\le C_5\sqrt{2}h^k w_h(t) + C_4\|\boldsymbol{\Psi}_h\|^2_{\varepsilon_0}\\
&\le C_5\sqrt{2}h^k w_h(t) + C_4 w_h^2(t).
\end{aligned}
$$

Integrating this inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\|\boldsymbol{\Psi}_h(t)\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{1}{2}\|\boldsymbol{\Phi}_h(t)\|^2_{\mu_0} \\
&+ \frac{\varepsilon_0}{2}\int_\Omega \chi^{(3)}\left[|\mathbf{E}(t)|^2|\boldsymbol{\Psi}_h(t)|^2 + 2|\mathbf{E}(t)^\top\boldsymbol{\Psi}_h(t)|^2\right] d\mathbf{x} \\
&\le \frac{1}{2}\|\boldsymbol{\Psi}_h(0)\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{1}{2}\|\boldsymbol{\Phi}_h(0)\|^2_{\mu_0} \\
&+ \frac{\varepsilon_0}{2}\int_\Omega \chi^{(3)}\left[|\mathbf{E}(0)|^2|\boldsymbol{\Psi}_h(0)|^2 + 2|\mathbf{E}(0)^\top\boldsymbol{\Psi}_h(0)|^2\right] d\mathbf{x} \\
&+ \int_0^t \left[C_5\sqrt{2}h^k w_h(s) + C_4 w_h^2(s)\right] ds. \qquad (3.50)
\end{aligned}
$$

By the monotonicity of the weighted norms w.r.t. the weight and the non-negativity of the integral term on the left-hand side, we see that

$$
\begin{aligned}
\frac{1}{2}w_h^2(t) &\le \frac{1}{2}\|\boldsymbol{\Psi}_h(t)\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{1}{2}\|\boldsymbol{\Phi}_h(t)\|^2_{\mu_0} \\
&+ \frac{\varepsilon_0}{2}\int_\Omega \chi^{(3)}\left[|\mathbf{E}(t)|^2|\boldsymbol{\Psi}_h(t)|^2 + 2|\mathbf{E}(t)^\top\boldsymbol{\Psi}_h(t)|^2\right] d\mathbf{x}. \qquad (3.51)
\end{aligned}
$$

On the other hand, we have the estimates

$$
\|\boldsymbol{\Psi}_h(0)\|^2_{\varepsilon_0(1+\chi^{(1)})} \le \|1+\chi^{(1)}\|_{L^\infty(\Omega)}\|\boldsymbol{\Psi}_h(0)\|^2_{\varepsilon_0} \le \|1+\chi^{(1)}\|_{L^\infty(\Omega)} w_h^2(0) \qquad (3.52)
$$

and

$$
\begin{aligned}
&\varepsilon_0\int_\Omega \chi^{(3)}\left[|\mathbf{E}(0)|^2|\boldsymbol{\Psi}_h(0)|^2 + 2|\mathbf{E}(0)^\top\boldsymbol{\Psi}_h(0)|^2\right] d\mathbf{x} \\
&\le 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}(0)\|^2_{L^\infty(\Omega)}\|\boldsymbol{\Psi}_h(0)\|^2_{\varepsilon_0} \\
&\le 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}(0)\|^2_{L^\infty(\Omega)} w_h^2(0). \qquad (3.53)
\end{aligned}
$$

Combining (3.51), (3.52), (3.53) with (3.50), we get

$$
\begin{aligned}
\frac{1}{2}w_h^2(t) &\le \frac{1}{2}\|1+\chi^{(1)}\|_{L^\infty(\Omega)} w_h^2(0) + \frac{3}{2}\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}(0)\|^2_{L^\infty(\Omega)} w_h^2(0) \\
&+ \int_0^t \left[C_5\sqrt{2}h^k w_h(s) + C_4 w_h^2(s)\right] ds,
\end{aligned}
$$

or, equivalently,

$$
w_h^2(t) \le C_6^2 w_h^2(0) + \int_0^t \left[2C_5\sqrt{2}h^k w_h(s) + 2C_4 w_h^2(s)\right] ds, \qquad (3.54)
$$

where $C_6^2 := \|1 + \chi^{(1)}\|_{L^\infty(\Omega)} + 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}(0)\|^2_{L^\infty(\Omega)}$.

In the paper [39], a Gronwall-type lemma (Lemma 4.1) is specified which extracts a bound for the value $w(T)$ if an inequality like (3.54) is satisfied:

$$w_h(T) \leq C_6 e^{C_4 T} w_h(0) + C_5 \sqrt{2} h^k T e^{C_4 T}.$$

From this and the triangle inequality in conjunction with (1.10) and (1.13) the statement follows. $\square$

3.4 Time Discretization for Nonlinear Maxwell's Equations

In this section, we present a novel fully discrete scheme for the nonlinear Maxwell's equations. Our particular interest is to demonstrate that the time discretization by means of the classical backward Euler-type method satisfies a discrete energy estimate, is unconditionally stable and convergent even in the presence of cubic nonlinearities. Analogous investigations for the linear case (that is $\chi^{(3)} = 0$) have been presented in [12]. The time discretization considered here can be used not only in conjunction with the Lee-Madson scheme or the Nédeléc and Raviart-Thomas spatial discretizations, but also with other types of spatial discretizations. The Newton's method is often employed to obtain the unknown values $\mathbf{E}_h^n$ and $\mathbf{H}_h^n$ from the nonlinear equations (3.55)–(3.57) or (3.58)–(3.60).

We divide the time interval $(0, T)$ into $N \in \mathbb{N}$ equally spaced subintervals by using the nodal points

$$0 =: t^0 < t^1 < t^2 < \ldots < t^N := T,$$

with $t^n = n\Delta t$, $n = 0, 1, 2, \ldots, N$.

The Fully Discrete Scheme for the Lee-Madson Formulation

Given initial values $(\mathbf{E}_h^0, \mathbf{H}_h^0) \in \mathbf{W}_h \times \mathbf{U}_h$ of the approximate electric and magnetic field intensities, the fully discrete electric and magnetic field intensities $(\mathbf{E}_h^n, \mathbf{H}_h^n) \in \mathbf{W}_h \times \mathbf{U}_h$, $n = 1, 2, \ldots, N$, satisfy

$$\Big(\frac{\mathbf{D}_h^n - \mathbf{D}_h^{n-1}}{\Delta t}, \mathbf{\Psi}_h\Big) - (\nabla \times \mathbf{H}_h^n, \mathbf{\Psi}_h) = (\mathbf{J}_h^n, \mathbf{\Psi}_h) \qquad \forall\, \mathbf{\Psi}_h \in \mathbf{W}_h, \tag{3.55}$$

$$\begin{aligned}\big(\mathbf{D}_h^n - \mathbf{D}_h^{n-1}, \mathbf{\Psi}_h\big) &= (\varepsilon_0(1 + \chi^{(1)})\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{\Psi}_h) \\ &+ \frac{1}{2}\varepsilon_0\chi^{(3)}(((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{\Psi}_h) \\ &+ \big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{\Psi}_h) \qquad \forall\, \mathbf{\Psi}_h \in \mathbf{W}_h,\end{aligned} \tag{3.56}$$

$$\Big(\mu_0\frac{\mathbf{H}_h^n - \mathbf{H}_h^{n-1}}{\Delta t}, \mathbf{\Phi}_h\Big) + (\mathbf{E}_h^n, \nabla \times \mathbf{\Phi}_h) = 0 \qquad \forall\, \mathbf{\Phi}_h \in \mathbf{U}_h. \tag{3.57}$$

Note that in this scheme the differences $\mathbf{D}_h^n - \mathbf{D}_h^{n-1}$ of the displacement approximations only play the role of auxiliary variables.

The Fully Discrete Scheme for the Nédeléc and Raviart-Thomas Formulation

Similarly to the Lee-Madson formulation, here we prescribe initial values $(\mathbf{E}_h^0, \mathbf{H}_h^0) \in \mathbf{U}_{0h} \times \mathbf{V}_h$ of the approximate electric and magnetic field intensities and determine the fully discrete electric and magnetic field intensities $(\mathbf{E}_h^n, \mathbf{H}_h^n) \in \mathbf{U}_{0h} \times \mathbf{V}_h$, $n = 1, 2, \ldots, N$, such that the following system is satisfied:

$$\Big(\frac{\mathbf{D}_h^n - \mathbf{D}_h^{n-1}}{\Delta t}, \mathbf{\Psi}_h\Big) - (\mathbf{H}_h^n, \nabla \times \mathbf{\Psi}_h) = (\mathbf{J}_h^n, \mathbf{\Psi}_h) \qquad \forall\, \mathbf{\Psi}_h \in \mathbf{U}_{0h}, \tag{3.58}$$

$$\begin{aligned}\big(\mathbf{D}_h^n - \mathbf{D}_h^{n-1}, \mathbf{\Psi}_h\big) &= (\varepsilon_0(1 + \chi^{(1)})\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{\Psi}_h) \\ &+ \frac{1}{2}\varepsilon_0\chi^{(3)}(((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{\Psi}_h) \\ &+ \big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{\Psi}_h) \qquad \forall\, \mathbf{\Psi}_h \in \mathbf{U}_{0h},\end{aligned} \tag{3.59}$$

$$\Big(\mu_0\frac{\mathbf{H}_h^n - \mathbf{H}_h^{n-1}}{\Delta t}, \mathbf{\Phi}_h\Big) + (\nabla \times \mathbf{E}_h^n, \mathbf{\Phi}_h) = 0 \qquad \forall\, \mathbf{\Phi}_h \in \mathbf{V}_h. \tag{3.60}$$

As above, the differences $\mathbf{D}_h^n - \mathbf{D}_h^{n-1}$ play the role of auxiliary variables.

3.4.1 The Nonlinear Electromagnetic Energy at the Fully Discrete Level

The nonlinear electromagnetic energy for the fully discrete approximation (i.e. both in space and time) of the systems (3.55)–(3.57) and (3.58)–(3.60) at t^n, $n = 0, 1, 2, \ldots, N$, is defined by

$$\mathbf{Energy}_h^n := \|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^n)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^n\|^2_{\mu_0}. \tag{3.61}$$

In analogy to the boundedness results for the continuous and semi-discrete nonlinear electromagnetic energy (Thms. 3.3, 3.5), in this section we will show that the fully discrete nonlinear electromagnetic energy of the systems (3.55)–(3.57) and (3.58)–(3.60) at the final time step N is bounded, too.

Theorem 3.8 *Let $(\mathbf{E}_h^n, \mathbf{H}_h^n)$ be the fully discrete solution of (3.58)–(3.60). Then, for sufficiently small Δt and h, there exists a constant $C > 0$ independent of Δt and h such that*

$$\mathbf{Energy}_h^N = \|\mathbf{E}_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^N)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^N\|^2_{\mu_0} \leq C.$$

Remark 3.9 An analogous result can be obtained for the system (3.55)–(3.57).

Proof: Taking $\mathbf{\Psi}_h = 2\,\mathbf{E}_h^n$ in the equation (3.59), we have

$$\begin{aligned}\big(\mathbf{D}_h^n - \mathbf{D}_h^{n-1}, 2\,\mathbf{E}_h^n\big) &= 2\Big[(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n)\\ &+ \frac{1}{2}\varepsilon_0\chi^{(3)}(((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n)\\ &+ \big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n\big)\Big]. \end{aligned} \tag{3.62}$$

Taking $\mathbf{\Psi}_h = 2\Delta t\,\mathbf{E}_h^n$ in the equations (3.58), we see that

$$\big(\mathbf{D}_h^n - \mathbf{D}_h^{n-1}, 2\,\mathbf{E}_h^n\big) = 2\Delta t(\mathbf{H}_h^n, \nabla\times\mathbf{E}_h^n) + 2\Delta t(\mathbf{J}_h^n, \mathbf{E}_h^n). \tag{3.63}$$

Replacing the left-hand side of equation (3.62) by (3.63), we have that

$$\begin{aligned} &2\Big[(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n)\\ &+ \frac{1}{2}\varepsilon_0\chi^{(3)}(((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n)\\ &+ \big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n\big)\Big]\\ &- 2\Delta t(\mathbf{H}_h^n, \nabla\times\mathbf{E}_h^n) = 2\Delta t(\mathbf{J}_h^n, \mathbf{E}_h^n). \end{aligned} \tag{3.64}$$

Taking $\mathbf{\Phi}_h = 2\Delta t\mathbf{H}_h^n$ in the equation (3.60), we obtain

$$2(\mu_0(\mathbf{H}_h^n - \mathbf{H}_h^{n-1}), \mathbf{H}_h^n) + 2\Delta t\,(\nabla\times\mathbf{E}_h^n, \mathbf{H}_h^n) = 0. \tag{3.65}$$

Adding the equations (3.64) and (3.65), we see that

$$\begin{aligned} &2\Big[(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n) + (\mu_0(\mathbf{H}_h^n - \mathbf{H}_h^{n-1}), \mathbf{H}_h^n)\\ &+ \varepsilon_0\chi^{(3)}(\frac{1}{2}((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n)\\ &+ \big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n\big)\Big]\\ &- 2\Delta t(\mathbf{H}_h^n, \nabla\times\mathbf{E}_h^n) + 2\Delta t\,(\nabla\times\mathbf{E}_h^n, \mathbf{H}_h^n) = 2\Delta t(\mathbf{J}_h^n, \mathbf{E}_h^n). \end{aligned}$$

This implies

$$
\begin{aligned}
&2\Big[(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)+(\mu_0(\mathbf{H}_h^n-\mathbf{H}_h^{n-1}),\mathbf{H}_h^n)\\
&+\varepsilon_0\chi^{(3)}(\frac{1}{2}((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)\\
&+\big(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right]\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n\big)\Big]\\
&=2\Delta t(\mathbf{J}_h^n,\mathbf{E}_h^n).
\end{aligned}
\tag{3.66}
$$

Now the estimate (1) from Lemma 5.1 is applied to the first and second terms on the left-hand side. Then, the first term from the left-hand side of the equation (3.66) can be written an estimated as

$$
\begin{aligned}
&2(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n)\\
&=\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})}-\|\mathbf{E}_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})}\\
&\geq\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}-\|\mathbf{E}_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})}.
\end{aligned}
$$

The second term from the left-hand side of the equation (3.66) is estimated in a similar way:

$$
\begin{aligned}
2(\mu_0(\mathbf{H}_h^n-\mathbf{H}_h^{n-1}),\mathbf{H}_h^n)=&\|\mathbf{H}_h^n\|^2_{\mu_0}+\|\mathbf{H}_h^n-\mathbf{H}_h^{n-1}\|^2_{\mu_0}-\|\mathbf{H}_h^{n-1}\|^2_{\mu_0}\\
\geq&\|\mathbf{H}_h^n\|^2_{\mu_0}-\|\mathbf{H}_h^{n-1}\|^2_{\mu_0}.
\end{aligned}
$$

The third and the fourth terms from the left-hand side of equation (3.66) can be treated as follows. Writing the test function $\mathbf{E}_h^n$ in the form

$$
\mathbf{E}_h^n=\frac{1}{2}\big(\mathbf{E}_h^n+\mathbf{E}_h^{n-1}\big)+\frac{1}{2}\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),
$$

we have that

$$
\begin{aligned}
&\Big(\varepsilon_0\chi^{(3)}\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n\Big)\\
&=\frac{1}{4}\Big(\varepsilon_0\chi^{(3)}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n+\mathbf{E}_h^{n-1}\Big)\\
&\quad+\frac{1}{4}\Big(\varepsilon_0\chi^{(3)}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\Big)\\
&=\frac{1}{4}\Big(\varepsilon_0\chi^{(3)}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\mathbf{E}_h^n,\mathbf{E}_h^n\Big)\\
&\quad-\frac{1}{4}\Big(\varepsilon_0\chi^{(3)}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\mathbf{E}_h^{n-1},\mathbf{E}_h^{n-1}\Big)\\
&\quad+\frac{1}{4}\Big(\varepsilon_0\chi^{(3)}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\Big)\\
&=\frac{1}{4}\Big(\varepsilon_0\chi^{(3)}(\mathbf{E}_h^n)^2\mathbf{E}_h^n,\mathbf{E}_h^n\Big)-\frac{1}{4}\Big(\varepsilon_0\chi^{(3)}(\mathbf{E}_h^{n-1})^2\mathbf{E}_h^{n-1},\mathbf{E}_h^{n-1}\Big)\\
&\quad+\frac{1}{4}\Big(\varepsilon_0\chi^{(3)}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\Big)\\
&\geq\frac{1}{4}\big\|(\mathbf{E}_h^n)^2\big\|^2_{\varepsilon_0\chi^{(3)}}-\frac{1}{4}\big\|(\mathbf{E}_h^{n-1})^2\big\|^2_{\varepsilon_0\chi^{(3)}}.
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
&\Big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\Big]\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n\Big)\\
&=\frac{1}{2}\Big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\Big]\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n+\mathbf{E}_h^{n-1}\Big)\\
&\quad+\frac{1}{2}\Big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\Big]\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\Big)\\
&=\frac{1}{2}\Big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\Big]\mathbf{E}_h^n,\mathbf{E}_h^n\Big)\\
&\quad-\frac{1}{2}\Big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\Big]\mathbf{E}_h^{n-1},\mathbf{E}_h^{n-1}\Big)\\
&\quad+\frac{1}{2}\Big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\Big]\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\Big)\\
&=\frac{1}{2}\Big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T\Big]\mathbf{E}_h^n,\mathbf{E}_h^n\Big)-\frac{1}{2}\Big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\Big]\mathbf{E}_h^{n-1},\mathbf{E}_h^{n-1}\Big)\\
&\quad+\frac{1}{2}\Big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\Big]\big(\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\big),\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\Big)\\
&\geq\frac{1}{2}\big\|(\mathbf{E}_h^n)^2\big\|^2_{\varepsilon_0\chi^{(3)}}-\frac{1}{2}\big\|(\mathbf{E}_h^{n-1})^2\big\|^2_{\varepsilon_0\chi^{(3)}}.
\end{aligned}
$$

So the left-hand side of the equation (3.66) can be estimated as follows:

$$
\begin{aligned}
&\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\mathbf{E}_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^n)^2\|^2_{\varepsilon_0\chi^{(3)}} - \frac{3}{2}\|(\mathbf{E}_h^{n-1})^2\|^2_{\varepsilon_0\chi^{(3)}} \\
&+ \|\mathbf{H}_h^n\|^2_{\mu_0} - \|\mathbf{H}_h^{n-1}\|^2_{\mu_0} \\
&\leq 2\Big[(\varepsilon_0(1+\chi^{(1)})\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n) \\
&+ \varepsilon_0\chi^{(3)}(\frac{1}{2}((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2)\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n) \\
&+ \big(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\Big]\,\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\big), \mathbf{E}_h^n) \\
&+ (\mu_0(\mathbf{H}_h^n - \mathbf{H}_h^{n-1}), \mathbf{H}_h^n)\Big].
\end{aligned} \tag{3.67}
$$

The right-hand side of the equation (3.66) is estimated by means of the inequality (2) from Lemma 5.1. This gives

$$
\begin{aligned}
2\Delta t\,(\mathbf{J}_h^n, \mathbf{E}_h^n) &= \Delta t\,([\varepsilon_0(1+\chi^{(1)})]^{-1/2}\mathbf{J}_h^n, [\varepsilon_0(1+\chi^{(1)})]^{1/2}\mathbf{E}_h^n) \\
&\leq \Delta t\,\|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \Delta t\,\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.
\end{aligned}
$$

Finally, using this estimate together with (3.67) in (3.66), we get

$$
\begin{aligned}
&\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\mathbf{E}_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^n)^2\|^2_{\varepsilon_0\chi^{(3)}} - \frac{3}{2}\|(\mathbf{E}_h^{n-1})^2\|^2_{\varepsilon_0\chi^{(3)}} \\
&+ \|\mathbf{H}_h^n\|^2_{\mu_0} - \|\mathbf{H}_h^{n-1}\|^2_{\mu_0} \leq \Delta t\,\|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \Delta t\,\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.
\end{aligned}
$$

Summing up from $n = 1$ to N, we arrive at

$$
\begin{aligned}
&\|\mathbf{E}_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\mathbf{E}_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^N)^2\|^2_{\varepsilon_0\chi^{(3)}} - \frac{3}{2}\|(\mathbf{E}_h^0)^2\|^2_{\varepsilon_0\chi^{(3)}} \\
&+ \|\mathbf{H}_h^N\|^2_{\mu_0} - \|\mathbf{H}_h^0\|^2_{\mu_0} \\
&\leq \sum_{n=1}^{N}\Delta t\,\|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \sum_{n=1}^{N}\Delta t\,\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.
\end{aligned} \tag{3.68}
$$

Therefore, we have

$$
\begin{aligned}
&\|\mathbf{E}_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^N)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^N\|^2_{\mu_0} \\
&\leq \left[\|\mathbf{E}_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^0)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^0\|^2_{\mu_0}\right] \\
&\quad + \sum_{n=1}^{N} \Delta t\, \|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \sum_{n=1}^{N} \Delta t\, \|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} \\
&\leq \Delta t \sum_{n=1}^{N} \left[\|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^n)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^n\|^2_{\mu_0}\right] \\
&\quad + \Delta t \sum_{n=1}^{N} \|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} \\
&\quad + \|\mathbf{E}_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^0)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^0\|^2_{\mu_0}.
\end{aligned}
$$

Now we employ the Gronwall's inequality (Lemma 5.3) with

$\delta := \Delta t \geq 0$,
$g_0 := \|\mathbf{E}_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^0)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^0\|^2_{\mu_0} \geq 0$,
$a_n := \|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^n)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^n\|^2_{\mu_0} \geq 0$,
$b_n := 0$,
$c_0 := 0$, $c_n := \|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} \geq 0$ for $n \in \mathbb{N}$, and
$\gamma_0 := 0$, $\gamma_n := 1 \geq 0$ for $n \in \mathbb{N}$.

Then the condition $\gamma_n\,\delta < 1$ corresponds to $\Delta t < 1$, and with it we get

$$
\begin{aligned}
&\|\mathbf{E}_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^N)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^N\|^2_{\mu_0} \\
&\leq \left(\Delta t \sum_{n=1}^{N} \|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \|\mathbf{E}_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^0)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^0\|^2_{\mu_0}\right) \\
&\quad \times \exp\left(\Delta t \sum_{n=1}^{N} (1-\Delta t)^{-1}\right).
\end{aligned}
$$

For $\Delta t \leq \frac{1}{2}$, this leads to

$$
\begin{aligned}
&\|\mathbf{E}_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^N)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^N\|^2_{\mu_0} \\
&\leq \left(\Delta t \sum_{n=1}^{N} \|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \|\mathbf{E}_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{3}{2}\|(\mathbf{E}_h^0)^2\|^2_{\varepsilon_0\chi^{(3)}} + \|\mathbf{H}_h^0\|^2_{\mu_0}\right) \\
&\quad \times \exp(2T).
\end{aligned}
$$

Since the term $\Delta t \sum_{n=1}^{N} \|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}}$ is an approximation to

$$\int_0^T \|\mathbf{J}_h^n\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} d\tau = \|\mathbf{J}_h^n\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}}(\Omega))},$$

it is bounded. $\square$

In what follows we will make use of different variants for the representation of terms like $\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})$. To this purpose we remember the Newton-Leibniz formula:

$$\mathbf{u}(t) = \mathbf{u}(t^{n-1}) + \int_{t^{n-1}}^{t} \partial_t \mathbf{u}(s) ds \quad \text{for all } \mathbf{u} \in \mathbf{C}^1(0,T,X),$$

where, as in Section 1.1 (Chapter 1), X is a Banach space. In particular, for $t = t^n$ it holds that

$$\mathbf{u}(t^n) - \mathbf{u}(t^{n-1}) = \Delta t\, \mathbf{r}_{\mathbf{u}}^n \quad \text{whith} \quad \mathbf{r}_{\mathbf{u}}^n := \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \partial_t \mathbf{u}(s) ds. \tag{3.69}$$

Furthermore, from Taylor's formula with integral remainder it follows that

$$\mathbf{u}(t) = \mathbf{u}(t^n) + \mathbf{u}_t(t^n)(t - t^n) + \int_{t^n}^{t} (t-s)\partial_{tt}\mathbf{u}(s) ds \quad \text{for all } \mathbf{u} \in \mathbf{C}^2(0,T,X).$$

Hence, with $t = t^{n-1}$ we have:

$$\frac{\mathbf{u}(t^n) - \mathbf{u}(t^{n-1})}{\Delta t} = \partial_t \mathbf{u}(t^n) + \mathbf{R}_{\mathbf{u}}^n, \tag{3.70}$$

where

$$\mathbf{R}_{\mathbf{u}}^n := \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} (t^{n-1} - s)\partial_{tt}\mathbf{u}(s) ds. \tag{3.71}$$

The remainder terms $\mathbf{r}_{\mathbf{u}}^n$, $\mathbf{R}_{\mathbf{u}}^n$ allow the following estimates.

Lemma 3.10 *Let X be a Banach space with the norm $\|\cdot\|_X$. The following estimates hold:*

1. $\|\mathbf{r}_{\mathbf{u}}^n\|_X \le \sqrt{\frac{1}{\Delta t}}\, \|\partial_t \mathbf{u}\|_{\mathbf{L}^2(t^{n-1},t^n,X)}, \quad \mathbf{u} \in \mathbf{C}^1(t^{n-1}, t^n, X),$
2. $\sum_{n=1}^{N} \|\mathbf{r}_{\mathbf{u}}^n\|_X^2 \le \frac{1}{\Delta t} \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(0,T,X)}, \quad \mathbf{u} \in \mathbf{C}^1(0,T,X),$
3. $\|\mathbf{R}_{\mathbf{u}}^n\|_X \le \sqrt{\frac{\Delta t}{3}}\, \|\partial_{tt} \mathbf{u}\|_{\mathbf{L}^2(t^{n-1},t^n,X)}, \quad \mathbf{u} \in \mathbf{C}^2(t^{n-1}, t^n, X),$

4. $\sum_{n=1}^{N} \|\mathbf{R}_{\mathbf{u}}^n\|_X^2 \leq \frac{\Delta t}{3} \|\partial_{tt}\mathbf{u}\|^2_{\mathbf{L}^2(0,T,X)}, \quad \mathbf{u} \in \mathbf{C}^2(0,T,X).$

Proof: (1) By the definition (3.69), we have

$$\|\mathbf{r}_{\mathbf{u}}^n\|_X = \frac{1}{\Delta t}\left\|\int_{t^{n-1}}^{t^n} \partial_t \mathbf{u}(s) ds\right\|_X \leq \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} \|\partial_t \mathbf{u}(s)\|_X ds$$
$$\leq \frac{1}{\Delta t}\left\{\int_{t^{n-1}}^{t^n} 1\, ds\right\}^{1/2}\left\{\int_{t^{n-1}}^{t^n} \|\partial_t \mathbf{u}(s)\|_X^2 ds\right\}^{1/2} = \sqrt{\frac{1}{\Delta t}}\, \|\partial_t \mathbf{u}\|_{\mathbf{L}^2(t^{n-1},t^n,X)}.$$

(2) is a simple consequence of (1) and an elementar integral property:

$$\sum_{n=1}^{N} \|\mathbf{r}_{\mathbf{u}}^n\|_X^2 \leq \frac{1}{\Delta t}\sum_{n=1}^{N} \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(t^{n-1},t^n,X)} = \frac{1}{\Delta t}\, \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(0,T,X)}.$$

(3) From(3.71), we have

$$\begin{aligned}\|\mathbf{R}_{\mathbf{u}}^n\|_X &= \frac{1}{\Delta t}\left\|\int_{t^{n-1}}^{t^n} (t^{n-1} - s)\partial_{tt}\mathbf{u}(s) ds\right\|_X \\ &\leq \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} (s - t^{n-1})\|\partial_{tt}\mathbf{u}(s)\|_X ds \\ &\leq \frac{1}{\Delta t}\left\{\int_{t^{n-1}}^{t^n} (s - t^{n-1})^2 ds\right\}^{1/2}\left\{\int_{t^{n-1}}^{t^n} \|\partial_{tt}\mathbf{u}(s)\|_X^2 ds\right\}^{1/2} \\ &= \sqrt{\frac{\Delta t}{3}}\, \|\partial_{tt}\mathbf{u}\|_{\mathbf{L}^2(t^{n-1},t^n,X)}.\end{aligned}$$

(4) This proof is analogous to the proof of (2). □

Before formulating the fully discrete theorem, we introduce the error terms for the electric field as

$$\zeta^n := \mathbf{E}(t^n) - \mathbf{E}_h^n = \eta^n - \eta_h^n, \tag{3.72}$$

where

$$\eta^n := \mathbf{E}(t^n) - \tilde{\mathbf{P}}_h\mathbf{E}(t^n), \quad \eta_h^n := \mathbf{E}_h^n - \tilde{\mathbf{P}}_h\mathbf{E}(t^n). \tag{3.73}$$

Analogously, for the magnetic field we set

$$\xi^n := \mathbf{H}(t^n) - \mathbf{H}_h^n = \theta^n - \theta_h^n, \tag{3.74}$$

where

$$\theta^n := \mathbf{H}(t^n) - \Pi_{1h}\mathbf{H}(t^n), \quad \theta_h^n := \mathbf{H}_h^n - \Pi_{1h}\mathbf{H}(t^n). \tag{3.75}$$

Finally, we denote the discrete time derivative of the sequence $(\mathbf{E}_h^n)$ at t^n by

$$\partial_{\Delta t}\mathbf{E}_h^n := \frac{1}{\Delta t}\left[\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\right]. \tag{3.76}$$

3.5 Error Estimates for the Fully Discrete Nonlinear Problem

Theorem 3.11 *Suppose additionally* $\chi^{(1)}$, $\chi^{(3)} \in L^\infty(\Omega)$. *Let* $(\mathbf{E}, \mathbf{H})$ *be the solution of (3.11)–(3.13) with* $\mathbf{J} = 0$ *such that, for some* $k \in \mathbb{N}$,

$$
\begin{aligned}
&\mathbf{E} \in \mathbf{C}^1(0, T, \mathbf{L}^\infty(\Omega) \cap \mathbf{H}^k(\Omega)), \ \partial_{tt}\mathbf{E} \in \mathbf{L}^2(0, T, \mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega)), \\
&\mathbf{H} \in \mathbf{C}^1(0, T, \mathbf{H}^{k+1}(\Omega)), \ \partial_{tt}\mathbf{H} \in \mathbf{L}^2(0, T, \mathbf{L}^2_{\mu_0}(\Omega)),
\end{aligned}
$$

and let $(\mathbf{E}_h^n, \mathbf{H}_h^n)$ *be the fully discrete solution of (3.55)–(3.57) such that there is a constant* $C^* > 0$ *independent of* Δt *and* h *such that* $\|\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega))} \le C^*$ *and* $\|\partial_{\Delta t}\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega))} \le C^*$ *for all* $n = 1, 2, \ldots, N$. *Then, for sufficiently small* Δt *and* h, *the following error estimate holds:*

$$
\|\mathbf{E}(t^N) - \mathbf{E}_h^N\|_{\varepsilon_0(1+\chi^{(1)})} + \|\mathbf{H}(t^N) - \mathbf{H}_h^N\|_{\mu_0} \le C\left[h^k + \Delta t\right],
$$

where the constant $C > 0$ *does not depend on* Δt *and* h *(the structure of* C *will be seen from the proof).*

Proof: Eliminating in the equations (3.55)–(3.56) the difference term $\mathbf{D}_h^n - \mathbf{D}_h^{n-1}$, we obtain

$$
\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})\Big(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h) - (\nabla \times \mathbf{H}_h^n, \mathbf{\Psi}_h) \\
&\quad + (\frac{1}{2}\varepsilon_0\chi^{(3)}((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2)\Big(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h) \\
&\quad + \big(\varepsilon_0\chi^{(3)}\left(\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right)\left(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h\right) = 0.
\end{aligned}
\tag{3.77}
$$

Taking $\mathbf{\Psi} = \mathbf{\Psi}_h$ and $t = t^n$ in the equations (3.11)–(3.12) and replacing the term $\partial_t\mathbf{E}(t^n)$ by means of (3.70), we have

$$
\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})\Big(\frac{\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h) - (\nabla \times \mathbf{H}(t^n), \mathbf{\Psi}_h) \\
&\quad + \big(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\Big(\frac{\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\big) \\
&\quad + \big(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\left(\frac{\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})}{\Delta t}\right), \mathbf{\Psi}_h\big) \\
&\quad = (\varepsilon_0(1+\chi^{(1)})\mathbf{R}_\mathbf{E}^n, \mathbf{\Psi}_h) + \big(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_\mathbf{E}^n, \mathbf{\Psi}_h\big) \\
&\quad + \big(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_\mathbf{E}^n, \mathbf{\Psi}_h\big).
\end{aligned}
\tag{3.78}
$$

Subtracting the equation (3.77) from the equation (3.78) and adding to both sides the two terms $\left(\frac{1}{2}\varepsilon_0\chi^{(3)}\Big[(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\Big]\Big(\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\right)$ and $\left(\varepsilon_0\chi^{(3)}\Big[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\Big]\Big(\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\right)$, we obtain

$$
\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})\Big(\frac{[\mathbf{E}(t^n)-\mathbf{E}_h^n]-[\mathbf{E}(t^{n-1})-\mathbf{E}_h^{n-1}]}{\Delta t}\Big), \mathbf{\Psi}_h)\\
&-(\nabla\times\big(\mathbf{H}(t^n)-\mathbf{H}_h^n\big), \mathbf{\Psi}_h)\\
&+\Big(\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\Big]\Big(\frac{[\mathbf{E}(t^n)-\mathbf{E}_h^n]-[\mathbf{E}(t^{n-1})-\mathbf{E}_h^{n-1}]}{\Delta t}\Big), \mathbf{\Psi}_h\Big)\\
&+\Big(\varepsilon_0\chi^{(3)}\big[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big]\Big(\frac{[\mathbf{E}(t^n)-\mathbf{E}_h^n]-[\mathbf{E}(t^{n-1})-\mathbf{E}_h^{n-1}]}{\Delta t}\Big), \mathbf{\Psi}_h\Big)\\
&=(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h)\\
&+\big(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h\big)+\big(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h\big)\\
&-\varepsilon_0\chi^{(3)}\Big[\Big(\Big[|\mathbf{E}(t^n)|^2-\frac{1}{2}[(\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2]\Big]\Big(\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\Big)\\
&-\Big(\big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T-\big(\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big)\big]\Big(\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\Big)\Big].
\end{aligned}
$$

Remembering the error terms $\zeta^n=\mathbf{E}(t^n)-\mathbf{E}_h^n$, $\xi^n=\mathbf{H}(t^n)-\mathbf{H}_h^n$ introduced in (3.72) and (3.74), we can write

$$
\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})\Big(\frac{\zeta^n-\zeta^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h)-(\nabla\times\xi^n, \mathbf{\Psi}_h)\\
&+\Big(\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\Big]\Big(\frac{\zeta^n-\zeta^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h\Big)\\
&+\Big(\varepsilon_0\chi^{(3)}\big[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big]\Big(\frac{\zeta^n-\zeta^{n-1}}{\Delta t}\Big), \mathbf{\Psi}_h\Big)\\
&=(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h)+\big(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h\big)\\
&+\big(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n, \mathbf{\Psi}_h\big)\\
&-\varepsilon_0\chi^{(3)}\Big[\Big(\Big[|\mathbf{E}(t^n)|^2-\frac{1}{2}[(\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2]\Big]\Big(\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\Big)\\
&-\Big(\big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T-\big(\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big)\big]\Big(\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}\Big), \mathbf{\Psi}_h\Big)\Big].
\end{aligned}
\tag{3.79}
$$

Taking $\mathbf{\Phi}=\mathbf{\Phi}_h$ and $t=t^n$ in the equation (3.13), subtracting the equation (3.57) from the result, and making use of (3.71), we get

$$
\begin{aligned}
&\Big(\mu_0\frac{[\mathbf{H}(t^n)-\mathbf{H}_h^n]-[\mathbf{H}(t^{n-1})-\mathbf{H}_h^{n-1}]}{\Delta t}, \mathbf{\Phi}_h\Big)+([\mathbf{E}(t^n)-\mathbf{E}_h^n], \nabla\times\mathbf{\Phi}_h)\\
&=(\mu_0\mathbf{R}_{\mathbf{H}}^n, \mathbf{\Phi}_h),
\end{aligned}
$$

or, in terms of the quantities defined in (3.72) and (3.74),

$$\Big(\mu_0\frac{\xi^n-\xi^{n-1}}{\Delta t},\boldsymbol{\Phi}_h\Big)+(\zeta^n,\nabla\times\boldsymbol{\Phi}_h)=(\mu_0\mathbf{R}_{\mathbf{H}}^n,\boldsymbol{\Phi}_h). \tag{3.80}$$

Using the decompositions $\zeta^n=\eta^n-\eta_h^n$, $\xi^n=\theta^n-\theta_h^n$ from (3.73) and (3.75), after a little rearrangement in the equations (3.79)–(3.80) we arrive at

$$\begin{aligned}
&(\varepsilon_0(1+\chi^{(1)})\Big(\frac{(\eta^n-\eta^{n-1})-(\eta_h^n-\eta_h^{n-1})}{\Delta t}\Big),\boldsymbol{\Psi}_h)-(\nabla\times(\theta^n-\theta_h^n),\boldsymbol{\Psi}_h)\\
&+\big(\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\Big]\Big(\frac{(\eta^n-\eta^{n-1})-(\eta_h^n-\eta_h^{n-1})}{\Delta t}\Big),\boldsymbol{\Psi}_h\big)\\
&+\big(\varepsilon_0\chi^{(3)}\big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\,\big]\,\Big(\frac{(\eta^n-\eta^{n-1})-(\eta_h^n-\eta_h^{n-1})}{\Delta t}\Big),\boldsymbol{\Psi}_h\big)\\
&=(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n,\boldsymbol{\Psi}_h)+\big(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n,\boldsymbol{\Psi}_h\big)\\
&+\big(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n,\boldsymbol{\Psi}_h\big)\\
&-\big(\varepsilon_0\chi^{(3)}\Big[|\mathbf{E}(t^n)|^2-\frac{1}{2}[(\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2]\Big]\Big(\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}\Big),\boldsymbol{\Psi}_h\big)\\
&-\big(\varepsilon_0\chi^{(3)}\big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T-\big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\,\big)\big]\\
&\cdot\Big(\frac{\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})}{\Delta t}\Big),\boldsymbol{\Psi}_h\big),
\end{aligned}$$

and

$$\Big(\mu_0\frac{(\theta^n-\theta^{n-1})-(\theta_h^n-\theta_h^{n-1})}{\Delta t},\boldsymbol{\Phi}_h\Big)+((\eta^n-\eta_h^n),\nabla\times\boldsymbol{\Phi}_h)=(\mu_0\mathbf{R}_{\mathbf{H}}^n,\boldsymbol{\Phi}_h).$$

Setting $\boldsymbol{\Psi}_h=2\Delta t\,\eta_h^n$ and $\boldsymbol{\Phi}_h=2\Delta t\theta_h^n$ in the above equations, we have

$$\begin{aligned}
&2\,(\varepsilon_0(1+\chi^{(1)})\big((\eta^n-\eta^{n-1})-(\eta_h^n-\eta_h^{n-1})\big),\eta_h^n)-(\nabla\times(\theta^n-\theta_h^n),2\Delta t\eta_h^n)\\
&+2\,\big(\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\Big]\Big((\eta^n-\eta^{n-1})-(\eta_h^n-\eta_h^{n-1})\Big),\eta_h^n\big)\\
&+2\,\big(\varepsilon_0\chi^{(3)}\big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\,\big]\,\big((\eta^n-\eta^{n-1})-(\eta_h^n-\eta_h^{n-1})\big),\eta_h^n\big)\\
&=(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n,2\Delta t\eta_h^n)+\ \big(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n,2\Delta t\,\eta_h^n\big)\\
&+\big(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n,2\Delta t\,\eta_h^n\big)\\
&-2\big(\varepsilon_0\chi^{(3)}\Big[|\mathbf{E}(t^n)|^2-\frac{1}{2}[(\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2]\Big]\big(\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})\big),\eta_h^n\big)\\
&-2\big(\varepsilon_0\chi^{(3)}\big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T-\big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\,\big)\big]\\
&\cdot\big(\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})\big),\eta_h^n\big),
\end{aligned}$$

and

$$\begin{aligned}
&2\,\big(\mu_0(\theta^n-\theta^{n-1})-(\theta_h^n-\theta_h^{n-1}),\theta_h^n\big)+2\Delta t(\eta^n-\eta_h^n,\nabla\times\theta_h^n)\\
&=(\mu_0\mathbf{R}_{\mathbf{H}}^n,2\Delta t\theta_h^n).
\end{aligned}$$

The above equations can be rearranged as

$$
\begin{aligned}
&2\left(\varepsilon_0(1+\chi^{(1)})(\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)-2\Delta t(\nabla\times\theta_h^n,\eta_h^n)\\
&\quad+2\left(\varepsilon_0\chi^{(3)}\left[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\right](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&\quad+2\left(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&\quad=2\left(\varepsilon_0(1+\chi^{(1)})(\eta^n-\eta^{n-1}),\eta_h^n\right)-2\Delta t(\nabla\times\theta^n,\eta_h^n)\\
&\quad-2\Delta t(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n,\,\eta_h^n)\\
&\quad+2\left(\varepsilon_0\chi^{(3)}\left[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\right](\eta^n-\eta^{n-1}),\eta_h^n\right)\\
&\quad+2\left(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right](\eta^n-\eta^{n-1}),\eta_h^n\right)\\
&\quad-2\Delta t\left(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n,\eta_h^n\right)-2\Delta t\left(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n,\eta_h^n\right)\\
&\quad+2\Big(\varepsilon_0\chi^{(3)}\left[|\mathbf{E}(t^n)|^2-\frac{1}{2}[(\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2]\right]\big(\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})\big),\eta_h^n\Big)\\
&\quad+2\Big(\varepsilon_0\chi^{(3)}\left[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T-\left(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right)\right]\\
&\quad\cdot\big(\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})\big),\eta_h^n\Big),
\end{aligned}
\tag{3.81}
$$

and

$$
\begin{aligned}
&2\left(\mu_0(\theta_h^n-\theta_h^{n-1}),\theta_h^n\right)+2\Delta t(\eta_h^n,\nabla\times\theta_h^n)=2\left(\mu_0(\theta^n-\theta^{n-1}),\theta_h^n\right)\\
&\quad+2\Delta t(\eta^n,\nabla\times\theta_h^n)-(\mu_0\mathbf{R}_{\mathbf{H}}^n,2\Delta t\theta_h^n).
\end{aligned}
\tag{3.82}
$$

The second terms from the left-hand sides of equations (3.81) and (3.82) vanish due to (1.11) and (1.8)–(1.9), respectively. Adding the equations

(3.81) and (3.82), we obtain

$$
\begin{aligned}
&2(\varepsilon_0(1+\chi^{(1)})(\eta_h^n-\eta_h^{n-1}),\eta_h^n)+2\left(\mu_0(\theta_h^n-\theta_h^{n-1}),\theta_h^n\right)\\
&\quad+2\left(\varepsilon_0\chi^{(3)}\left[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\right](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&\quad+2\left(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&\quad=2(\varepsilon_0(1+\chi^{(1)})(\eta^n-\eta^{n-1}),\eta_h^n)+2\big(\mu_0(\theta^n-\theta^{n-1}),\theta_h^n\big)\\
&\quad+2\left(\varepsilon_0\chi^{(3)}\left[\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big)\right](\eta^n-\eta^{n-1}),\eta_h^n\right)\\
&\quad+2\left(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right](\eta^n-\eta^{n-1}),\eta_h^n\right)\\
&\quad-2\Delta t(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n,\,\eta_h^n)-2\Delta t(\mu_0\mathbf{R}_{\mathbf{H}}^n,\theta_h^n)\\
&\quad-2\Delta t\left(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n,\eta_h^n\right)-2\Delta t\left(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n,\eta_h^n\right)\\
&\quad+2\varepsilon_0\chi^{(3)}\Big[\Big(\Big[|\mathbf{E}(t^n)|^2-\frac{1}{2}[(\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2]\Big]\big(\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})\big),\eta_h^n\Big)\\
&\quad+\Big(\big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T-\big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big)\big]\,\big(\mathbf{E}(t^n)-\mathbf{E}(t^{n-1})\big),\eta_h^n\Big)\Big].
\end{aligned}
\tag{3.83}
$$

An estimate of the left-hand side at level n

The identity (1) from Lemma 5.1 allows us to rewrite and estimate the first four terms on the left-hand side of (3.83) in the following way:

$$
\begin{aligned}
&2(\varepsilon_0(1+\chi^{(1)})(\eta_h^n-\eta_h^{n-1}),\eta_h^n)\\
&=\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\eta_h^n-\eta_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})}-\|\eta_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})}\\
&\geq\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}-\|\eta_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})},
\end{aligned}
$$

and

$$
\begin{aligned}
&2\left(\mu_0(\theta_h^n-\theta_h^{n-1}),\theta_h^n\right)\\
&=\|\theta_h^n\|^2_{\mu_0}+\|\theta_h^n-\theta_h^{n-1}\|^2_{\mu_0}-\|\theta_h^{n-1}\|^2_{\mu_0}\\
&\geq\|\theta_h^n\|^2_{\mu_0}-\|\theta_h^{n-1}\|^2_{\mu_0}.
\end{aligned}
$$

In order to simplify the treatment of the third and fourth terms, we introduce the abbreviations

$$
\mathbf{C}_1^{n-\frac{1}{2}}:=\frac{1}{2}\big((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\big),\tag{3.84}
$$

$$
\mathbf{C}_2^{n-\frac{1}{2}}:=\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T.\tag{3.85}
$$

Then we have that

$$
\begin{aligned}
&2\left(\varepsilon_0\chi^{(3)}\left[\frac{1}{2}\left((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\right)\right](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&=2\left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}(\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&=\left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^n,\eta_h^n\right)\\
&\quad+\left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}(\eta_h^n-\eta_h^{n-1}),\eta_h^n-\eta_h^{n-1}\right)-\left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^{n-1},\eta_h^{n-1}\right)\\
&\geq\left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^n,\eta_h^n\right)-\left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^{n-1},\eta_h^{n-1}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&2\left(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&=2\left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}(\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&=\left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^n,\eta_h^n\right)\\
&\quad+\left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}(\eta_h^n-\eta_h^{n-1}),\eta_h^n-\eta_h^n\right)-\left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^{n-1},\eta_h^{n-1}\right)\\
&\geq\left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^n,\eta_h^n\right)-\left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^{n-1},\eta_h^{n-1}\right).
\end{aligned}
$$

Here we have used the fact that the matrices $\mathbf{C}_2^{n-\frac{1}{2}}$, $n=1,2,\ldots,N$, are positively semidefinite.

In summary, the left-hand side of equation (3.83) can be estimated from below as follows:

$$
\begin{aligned}
&\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}-\|\eta_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\theta_h^n\|^2_{\mu_0}-\|\theta_h^{n-1}\|^2_{\mu_0}\\
&\quad+\left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^n,\eta_h^n\right)-\left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^{n-1},\eta_h^{n-1}\right)\\
&\quad+\left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^n,\eta_h^n\right)-\left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^{n-1},\eta_h^{n-1}\right)\\
&\leq 2(\varepsilon_0(1+\chi^{(1)})(\eta_h^n-\eta_h^{n-1}),\eta_h^n)+2\left(\mu_0(\theta_h^n-\theta_h^{n-1}),\theta_h^n\right)\\
&\quad+2\left(\varepsilon_0\chi^{(3)}\left[\frac{1}{2}\left((\mathbf{E}_h^n)^2+(\mathbf{E}_h^{n-1})^2\right)\right](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right)\\
&\quad+2\left(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\left[\mathbf{E}_h^n\right]^T+\mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right](\eta_h^n-\eta_h^{n-1}),\eta_h^n\right).
\end{aligned}
$$

So from equation (3.83) we get the inequality

$$
\begin{aligned}
&\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\eta_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^n\|^2_{\mu_0} - \|\theta_h^{n-1}\|^2_{\mu_0} \\
&\quad + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^n, \eta_h^n\right) - \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^{n-1}, \eta_h^{n-1}\right) \\
&\quad + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^n, \eta_h^n\right) - \left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^{n-1}, \eta_h^{n-1}\right) \\
&\leq 2(\varepsilon_0(1+\chi^{(1)})(\eta^n - \eta^{n-1}), \eta_h^n) + 2\big(\mu_0(\theta^n - \theta^{n-1}), \theta_h^n\big) \\
&\quad + 2\left(\varepsilon_0\chi^{(3)}\left[\frac{1}{2}\big((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\big)\right](\eta^n - \eta^{n-1}), \eta_h^n\right) \\
&\quad + 2\left(\varepsilon_0\chi^{(3)}\left[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\right](\eta^n - \eta^{n-1}), \eta_h^n\right) \\
&\quad - 2\Delta t(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_{\mathbf{E}}^n,\, \eta_h^n) - 2\Delta t(\mu_0\mathbf{R}_{\mathbf{H}}^n, \theta_h^n) \\
&\quad - 2\Delta t\left(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_{\mathbf{E}}^n, \eta_h^n\right) - 2\Delta t\left(\varepsilon_0\chi^{(3)}2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_{\mathbf{E}}^n, \eta_h^n\right) \\
&\quad + 2\varepsilon_0\chi^{(3)}\Big[\Big(\Big[|\mathbf{E}(t^n)|^2 - \frac{1}{2}[(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2]\Big]\big(\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})\big), \eta_h^n\Big) \\
&\quad + \Big(\Big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \Big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\Big)\Big]\big(\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})\big), \eta_h^n\Big)\Big].
\end{aligned}
$$

In order to simplify the further presentation, we denote the ten summands of the right-hand side in the specified order by $\tilde{\delta}_j^n$, $j = 1, \ldots, 10$ (the detailed definitions will be repeated later).

Now we sum up these inequalities from $n = 1$ to N:

$$
\begin{aligned}
&\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|^2_{\mu_0} - \|\theta_h^0\|^2_{\mu_0} \\
&\quad + \sum_{n=1}^{N}\left[\left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^n, \eta_h^n\right) - \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{n-\frac{1}{2}}\eta_h^{n-1}, \eta_h^{n-1}\right)\right] \\
&\quad + \sum_{n=1}^{N}\left[\left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^n, \eta_h^n\right) - \left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{n-\frac{1}{2}}\eta_h^{n-1}, \eta_h^{n-1}\right)\right] \\
&\leq \sum_{j=1}^{10}\tilde{\delta}_j,
\end{aligned}
$$

where

$$
\tilde{\delta}_j := \sum_{n=1}^{N}\tilde{\delta}_j^n, \quad j = 1, \ldots, 10. \tag{3.86}
$$

The application of Lemma 5.2 to the two sums on the left-hand side results in

$$
\begin{aligned}
&\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|^2_{\mu_0} - \|\theta_h^0\|^2_{\mu_0} \\
&\quad + \big(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{N-\frac{1}{2}}\eta_h^N, \eta_h^N\big) - \big(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{\frac{1}{2}}\eta_h^0, \eta_h^0\big) \\
&\quad + \sum_{n=1}^{N-1} \big(\varepsilon_0\chi^{(3)}\big[\mathbf{C}_1^{n-\frac{1}{2}} - \mathbf{C}_1^{n+\frac{1}{2}}\big]\eta_h^n, \eta_h^n\big) \\
&\quad + \big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{N-\frac{1}{2}}\eta_h^N, \eta_h^N\big) - \big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{\frac{1}{2}}\eta_h^0, \eta_h^0\big) \\
&\quad + \sum_{n=1}^{N-1} \big(\varepsilon_0\chi^{(3)}\big[\mathbf{C}_2^{n-\frac{1}{2}} - \mathbf{C}_2^{n+\frac{1}{2}}\big]\eta_h^n, \eta_h^n\big) \\
&\le \sum_{j=1}^{10} \tilde{\delta}_j.
\end{aligned}
$$

Setting

$$
\tilde{\delta}_{11}^n := \big(\varepsilon_0\chi^{(3)}\big[\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}}\big]\eta_h^n, \eta_h^n\big), \quad \tilde{\delta}_{12}^n := \big(\varepsilon_0\chi^{(3)}\big[\mathbf{C}_2^{n+\frac{1}{2}} - \mathbf{C}_2^{n-\frac{1}{2}}\big]\eta_h^n, \eta_h^n\big),
$$

we get

$$
\begin{aligned}
&\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|^2_{\mu_0} - \|\theta_h^0\|^2_{\mu_0} \\
&\quad + \big(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{N-\frac{1}{2}}\eta_h^N, \eta_h^N\big) - \big(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{\frac{1}{2}}\eta_h^0, \eta_h^0\big) \\
&\quad + \big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{N-\frac{1}{2}}\eta_h^N, \eta_h^N\big) - \big(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{\frac{1}{2}}\eta_h^0, \eta_h^0\big) \\
&\le \sum_{j=1}^{12} \tilde{\delta}_j,
\end{aligned}
\tag{3.87}
$$

where $\tilde{\delta}_{11}$, $\tilde{\delta}_{12}$ are defined in analogy to (3.86).

Estimation of the right-hand side

The first to fourth terms on the right-hand side of the inequality (3.87) are treated by means of the formula (3.69). Replacing there $\mathbf{u}$ by $(\mathbf{I} - \tilde{\mathbf{P}}_h)\mathbf{E}$

and $(\mathbf{I}-\Pi_{1h})\mathbf{H}$, respectively, we obtain for the the first term

$$\begin{aligned}
\tilde{\delta}_1^n &= 2\big(\varepsilon_0(1+\chi^{(1)})(\eta^n-\eta^{n-1}),\eta_h^n\big) = 2\Delta t\big(\varepsilon_0(1+\chi^{(1)})(\mathbf{I}-\tilde{\mathbf{P}}_h)\mathbf{r}_{\mathbf{E}}^n,\eta_h^n\big)\\
&= 2\Delta t\big(\varepsilon_0(1+\chi^{(1)})\mathbf{r}^n_{(\mathbf{I}-\tilde{\mathbf{P}}_h)\mathbf{E}},\eta_h^n\big)\\
&\le \Delta t\left[\|\mathbf{r}^n_{(\mathbf{I}-\tilde{\mathbf{P}}_h)\mathbf{E}}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\right]\\
&\qquad\qquad \text{(by Lemma 5.1(2))}\\
&\le \|\partial_t\big((\mathbf{I}-\tilde{\mathbf{P}}_h)\mathbf{E}\big)\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))}+\Delta t\,\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\\
&\qquad\qquad \text{(by Lemma 3.10(1))}\\
&\le C\varepsilon_0\|1+\chi^{(1)}\|_{L^\infty(\Omega)}\,h^{2k}\|\partial_t\mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{H}^k(\Omega))}+\Delta t\,\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\\
&\qquad\qquad \text{(cf. (3.48)).}
\end{aligned}$$

Thus we get

$$\tilde{\delta}_1=\sum_{n=1}^N\tilde{\delta}_1^n\le C\varepsilon_0\|1+\chi^{(1)}\|_{L^\infty(\Omega)}\,h^{2k}\|\partial_t\mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{H}^k(\Omega))}+\Delta t\,S_\eta^N, \tag{3.88}$$

where

$$S_\eta^N:=\sum_{n=1}^N\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.$$

Analogously, the second term on the right-hand side of the (3.87) can be written and estimated as

$$\begin{aligned}
\tilde{\delta}_2^n &= 2\big(\mu_0(\theta^n-\theta^{n-1}),\theta_h^n\big)=2\Delta t\big(\mu_0(\mathbf{I}-\Pi_{1h})\mathbf{r}^n_{\mathbf{H}},\theta_h^n\big)\\
&= 2\Delta t\big(\mu_0\mathbf{r}^n_{(\mathbf{I}-\Pi_{1h})\mathbf{H}},\theta_h^n\big)\\
&\le \Delta t\left[\|\mathbf{r}^n_{(\mathbf{I}-\Pi_{1h})\mathbf{H}}\|^2_{\mu_0}+\|\theta_h^n\|^2_{\mu_0}\right]\\
&\le C\mu_0h^{2k}\|\partial_t\mathbf{H}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{H}^{k+1}(\Omega))}+\Delta t\,\|\theta_h^n\|^2_{\mu_0},
\end{aligned}$$

where we have used (3.69), Lemma 3.10(1) and (3.49). Hence

$$\tilde{\delta}_2\le C\mu_0h^{2k}\|\partial_t\mathbf{H}\|^2_{\mathbf{L}^2(0,T,\mathbf{H}^{k+1}(\Omega))}+\Delta t\,S_\theta^N, \tag{3.89}$$

where

$$S_\theta^N:=\sum_{n=1}^N\|\theta_h^n\|^2_{\mu_0}.$$

The third term from the right-hand side of the inequality (3.87) is estimated as

$$\begin{aligned}
\tilde{\delta}_3^n &= \Big(\varepsilon_0\chi^{(3)}\Big[(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\Big](\eta^n - \eta^{n-1}), \eta_h^n\Big) \\
&= \Delta t\big(\varepsilon_0\chi^{(3)}\big[(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\big](\mathbf{I} - \tilde{\mathbf{P}}_h)\mathbf{r}_{\mathbf{E}}^n, \eta_h^n\big) \\
&= \Delta t\big(\varepsilon_0\chi^{(3)}\big[(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\big]\mathbf{r}^n_{(\mathbf{I}-\tilde{\mathbf{P}}_h)\mathbf{E}}, \eta_h^n\big) \\
&\le \Delta t\,\big\|(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\big\|_{L^\infty(\Omega)}\|\mathbf{r}^n_{(\mathbf{I}-\tilde{\mathbf{P}}_h)\mathbf{E}}\|_{\varepsilon_0\chi^{(3)}}\|\eta_h^n\|_{\varepsilon_0\chi^{(3)})} \\
&\le \Delta t\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\left[\|\mathbf{r}^n_{(\mathbf{I}-\tilde{\mathbf{P}}_h)\mathbf{E}}\|^2_{\varepsilon_0\chi^{(3)}} + \|\eta_h^n\|^2_{\varepsilon_0\chi^{(3)})}\right],
\end{aligned}$$

where we have used the notation

$$\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))} := \max_{n=0,1,\ldots,N}\|\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega)}.$$

Since

$$\begin{aligned}
&\|\mathbf{r}^n_{(\mathbf{I}-\tilde{\mathbf{P}}_h)\mathbf{E}}\|^2_{\varepsilon_0\chi^{(3)}} \\
&\le \frac{1}{\Delta t}\|\partial_t\big((\mathbf{I} - \tilde{\mathbf{P}}_h)\mathbf{E}\big)\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} \\
&\le \frac{C}{\Delta t}\varepsilon_0\|\chi^{(3)}\|_{L^\infty(\Omega)}\, h^{2k}\|\partial_t\mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{H}^k(\Omega))}, \\
&\|\eta_h^n\|^2_{\varepsilon_0\chi^{(3)})} = (\varepsilon_0\chi^{(3)}\eta_h^n, \eta_h^n) \\
&= \left(\varepsilon_0(1+\chi^{(1)})\frac{\chi^{(3)}}{1+\chi^{(1)}}\eta_h^n, \eta_h^n\right) \\
&\le \|\chi^{(3)}\|_{L^\infty(\Omega)}\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})},
\end{aligned}$$

we arrive at

$$\begin{aligned}
\tilde{\delta}_3^n &\le C\varepsilon_0\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\, h^{2k}\|\partial_t\mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{H}^k(\Omega))} \\
&\quad + \|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Delta t\,\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.
\end{aligned}$$

This leads to

$$\begin{aligned}
\tilde{\delta}_3 &\le C\varepsilon_0\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\, h^{2k}\|\partial_t\mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{H}^k(\Omega))} \\
&\quad + \|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Delta t\, S_\eta^N. \qquad (3.90)
\end{aligned}$$

The fourth term from the right-hand side of (3.87) is treated in a similar manner:

$$\begin{aligned}
\tilde{\delta}_4^n &= 2\big(\varepsilon_0\chi^{(3)}\big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\,\big]\,(\eta^n - \eta^{n-1}), \eta_h^n\big) \\
&= 2\Delta t\big(\varepsilon_0\chi^{(3)}\big[\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\,\big[\mathbf{E}_h^{n-1}\big]^T\,\big]\mathbf{r}^n_{(\mathbf{I}-\tilde{\mathbf{P}}_h)\mathbf{E}}, \eta_h^n\big) \\
&\le 2\Delta t\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\left[\|\mathbf{r}^n_{(\mathbf{I}-\tilde{\mathbf{P}}_h)\mathbf{E}}\|^2_{\varepsilon_0\chi^{(3)}} + \|\eta_h^n\|^2_{\varepsilon_0\chi^{(3)})}\right],
\end{aligned}$$

and as in the estimation for $\tilde{\delta}_3$, this results in

$$\tilde{\delta}_4 \leq C\varepsilon_0 \|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\, h^{2k} \|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{H}^k(\Omega))}$$
$$+ 2\|\chi^{(3)}\|_{L^\infty(\Omega)} \|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))} \Delta t\, S_\eta^N. \tag{3.91}$$

Now we turn to the consideration of the terms $\tilde{\delta}_5^n$ to $\tilde{\delta}_8^n$ containing the remainders $\mathbf{R}_\mathbf{E}^n$, $\mathbf{R}_\mathbf{H}^n$. For $\tilde{\delta}_5^n$ we have:

$$\tilde{\delta}_5^n = -2\Delta t(\varepsilon_0(1+\chi^{(1)})\mathbf{R}_\mathbf{E}^n,\, \eta_h^n) \leq \Delta t\, \|\mathbf{R}_\mathbf{E}^n\|^2_{\varepsilon_0(1+\chi^{(1)})} + \Delta t\, \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.$$

Then Lemma 3.10(4) implies that

$$\tilde{\delta}_5 \leq \frac{(\Delta t)^2}{3} \|\partial_{tt}\mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} + \Delta t\, S_\eta^N. \tag{3.92}$$

A completely analogous argument shows that

$$\tilde{\delta}_6 \leq \frac{(\Delta t)^2}{3} \|\partial_{tt}\mathbf{H}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\mu_0}(\Omega))} + \Delta t\, S_\theta^N. \tag{3.93}$$

The estimate of $\tilde{\delta}_7^n$ runs as follows:

$$\begin{aligned}
\tilde{\delta}_7^n &= -2\Delta t\left(\varepsilon_0\chi^{(3)}|\mathbf{E}(t^n)|^2\mathbf{R}_\mathbf{E}^n, \eta_h^n\right) \\
&= -2\Delta t\left(\varepsilon_0(1+\chi^{(1)})\frac{\chi^{(3)}}{1+\chi^{(1)}}|\mathbf{E}(t^n)|^2\mathbf{R}_\mathbf{E}^n, \eta_h^n\right) \\
&\leq \|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}_h^n\|^2_{\mathbf{L}^\infty(\Omega)}\left[\Delta t\, \|\mathbf{R}_\mathbf{E}^n\|^2_{\varepsilon_0(1+\chi^{(3)})} + \Delta t\, \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\right].
\end{aligned}$$

Then we get, using Lemma 3.10(4) again, that

$$\tilde{\delta}_7 \leq \frac{(\Delta t)^2}{3}\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|\partial_{tt}\mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))}$$
$$+ \|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Delta t\, S_\eta^N. \tag{3.94}$$

For $\tilde{\delta}_8^n = -4\Delta t\left(\varepsilon_0\chi^{(3)}\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T\,\mathbf{R}_\mathbf{E}^n, \eta_h^n\right)$, it is easy to see that

$$\tilde{\delta}_8 \leq 2\frac{(\Delta t)^2}{3}\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|\partial_{tt}\mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))}$$
$$+ 2\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Delta t\, S_\eta^N \tag{3.95}$$

holds. The estimation technique for $\tilde{\delta}_9^n$ and $\tilde{\delta}_{10}^n$ is similar to that for $\tilde{\delta}_3^n$ and $\tilde{\delta}_4^n$ in the sense that it is based on the remainders $\mathbf{r}_\mathbf{E}^n$, $\mathbf{r}_\mathbf{H}^n$. Namely, for $\tilde{\delta}_9^n$ we have, by (3.69), that

$$\begin{aligned}
\tilde{\delta}_9^n &= 2\big(\varepsilon_0\chi^{(3)}\Big[|\mathbf{E}(t^n)|^2 - \frac{1}{2}[(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2]\Big]\big(\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})\big), \eta_h^n\big) \\
&= \Delta t\left(\varepsilon_0\chi^{(3)}\Big[2|\mathbf{E}(t^n)|^2 - [(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2]\Big]\mathbf{r}_\mathbf{E}^n, \eta_h^n\right).
\end{aligned}$$

Next we consider the term in the big square brackets (cf. (3.72)):

$$
\begin{aligned}
&2|\mathbf{E}(t^n)|^2 - [(\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2] \\
&= (\mathbf{E}(t^n))^2 - (\mathbf{E}_h^n)^2 + (\mathbf{E}(t^n))^2 - (\mathbf{E}_h^{n-1})^2 \\
&= [\mathbf{E}(t^n) + \mathbf{E}_h^n]^T[\mathbf{E}(t^n) - \mathbf{E}_h^n] + [\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T[\mathbf{E}(t^n) - \mathbf{E}_h^{n-1}] \\
&= [\mathbf{E}(t^n) + \mathbf{E}_h^n]^T[\eta^n - \eta_h^n] + [\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T[\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})] \\
&\quad + [\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T[\eta^{n-1} - \eta_h^{n-1}] \\
&= [\mathbf{E}(t^n) + \mathbf{E}_h^n]^T\eta^n - [\mathbf{E}(t^n) + \mathbf{E}_h^n]^T\eta_h^n + \Delta t\,[\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T\mathbf{r}_{\mathbf{E}}^n \\
&\quad + [\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T\eta^{n-1} - [\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T\eta_h^{n-1}.
\end{aligned}
$$

These five summands generate in a straightforward way a decomposition of $\tilde{\delta}_9^n$:

$$
\tilde{\delta}_9^n = \sum_{j=1}^{5} \tilde{\delta}_{9j}^n.
$$

The subsequent steps are devoted to the estimation of the five terms $\tilde{\delta}_{9j}^n$. We have that

$$
\begin{aligned}
\tilde{\delta}_{91}^n &= \Delta t\left(\varepsilon_0\chi^{(3)}[\mathbf{E}(t^n) + \mathbf{E}_h^n]^T\eta^n\mathbf{r}_{\mathbf{E}}^n, \eta_h^n\right) \\
&= \Delta t\left(\varepsilon_0(1+\chi^{(1)})\frac{\chi^{(3)}}{1+\chi^{(1)}}[\mathbf{E}(t^n) + \mathbf{E}_h^n]^T\eta^n\mathbf{r}_{\mathbf{E}}^n, \eta_h^n\right) \\
&\le \Delta t\,\|\chi^{(3)}\|_{L^\infty(\Omega)}\|[\mathbf{E}(t^n) + \mathbf{E}_h^n]^T\eta^n\|_{\varepsilon_0(1+\chi^{(1)})}\|\,[\mathbf{r}_{\mathbf{E}}^n]^T\,\eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})} \\
&\le \Delta t\,\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{E}(t^n) + \mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega)}\|\eta^n\|_{\varepsilon_0(1+\chi^{(1)})}\|\mathbf{r}_{\mathbf{E}}^n\|_{\mathbf{L}^\infty(\Omega)}\|\eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})} \\
&\le \Delta t\,\|\chi^{(3)}\|_{L^\infty(\Omega)}\left[\|\mathbf{E}(t^n)\|_{\mathbf{L}^\infty(\Omega)} + \|\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega)}\right] \\
&\cdot \|\mathbf{r}_{\mathbf{E}}^n\|_{\mathbf{L}^\infty(\Omega)}\|\eta^n\|_{\varepsilon_0(1+\chi^{(1)})}\|\eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\|\mathbf{E}(t^n)\|_{\mathbf{L}^\infty(\Omega)} + \|\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega)} &\le \|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}, \\
\|\mathbf{r}_{\mathbf{E}}^n\|_{\mathbf{L}^\infty(\Omega)} \le \sqrt{\frac{1}{\Delta t}}\,\|\partial_t\mathbf{E}\|_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^\infty(\Omega))} &\le \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))},
\end{aligned}
\tag{3.96}
$$

we obtain

$$\begin{aligned}
\tilde{\delta}_{91}^n &\le \Delta t\, \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\right] \\
&\quad \times \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))} \|\eta^n\|_{\varepsilon_0(1+\chi^{(1)})} \|\eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})} \\
&\le \frac{\Delta t}{2}\, \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\right] \\
&\quad \times \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))} \left[\|\eta^n\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\right] \\
&\le C\Delta t\, h^{2k}\, \varepsilon_0 \|1+\chi^{(1)}\|_{L^\infty(\Omega)} \|\chi^{(3)}\|_{L^\infty(\Omega)} \Big[\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{L}^\infty(\Omega))} \\
&+ \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Big] \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))} \|\mathbf{E}\|^2_{\mathbf{C}(0,T,\mathbf{H}^k(\Omega))} \\
&\quad + \frac{\Delta t}{2}\, \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\right] \\
&\quad \times \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))} \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} \\
&\qquad\qquad \text{(cf. (3.48))}.
\end{aligned}$$

The treatment of $\tilde{\delta}_{92}^n$ is quite similar to $\tilde{\delta}_{91}^n$:

$$\begin{aligned}
\tilde{\delta}_{92}^n &= -\Delta t \left(\varepsilon_0 \chi^{(3)} [\mathbf{E}(t^n) + \mathbf{E}_h^n]^T \eta_h^n \mathbf{r}_{\mathbf{E}}^n, \eta_h^n\right) \\
&= -\Delta t \left(\varepsilon_0 (1+\chi^{(1)}) \frac{\chi^{(3)}}{1+\chi^{(1)}} [\mathbf{E}(t^n) + \mathbf{E}_h^n]^T \eta_h^n \mathbf{r}_{\mathbf{E}}^n, \eta_h^n\right) \\
&\le \Delta t\, \|\chi^{(3)}\|_{L^\infty(\Omega)} \|[\mathbf{E}(t^n) + \mathbf{E}_h^n]^T \eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})} \|\, [\mathbf{r}_{\mathbf{E}}^n]^T\, \eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})} \\
&\le \Delta t\, \|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}(t^n) + \mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega)} \|\mathbf{r}_{\mathbf{E}}^n\|_{\mathbf{L}^\infty(\Omega)} \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} \\
&\le \Delta t\, \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}(t^n)\|_{\mathbf{L}^\infty(\Omega)} + \|\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega)}\right] \|\mathbf{r}_{\mathbf{E}}^n\|_{\mathbf{L}^\infty(\Omega)} \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} \\
&\le \Delta t\, \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\right] \\
&\cdot \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))} \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} \\
&\qquad\qquad \text{(by (3.96))}.
\end{aligned}$$

Next we see that

$$\begin{aligned}
\tilde{\delta}_{93}^n &= (\Delta t)^2 \big(\varepsilon_0 \chi^{(3)} [\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T \mathbf{r}_{\mathbf{E}}^n \mathbf{r}_{\mathbf{E}}^n, \eta_h^n\big) \\
&= (\Delta t)^2 \big(\varepsilon_0 (1+\chi^{(1)}) \frac{\chi^{(3)}}{1+\chi^{(1)}} [\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T \mathbf{r}_{\mathbf{E}}^n \mathbf{r}_{\mathbf{E}}^n, \eta_h^n\big) \\
&\le (\Delta t)^2 \|\chi^{(3)}\|_{L^\infty(\Omega)} \|[\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T \mathbf{r}_{\mathbf{E}}^n\|_{\varepsilon_0(1+\chi^{(1)})} \|\, [\mathbf{r}_{\mathbf{E}}^n]^T\, \eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})} \\
&\le (\Delta t)^2 \|\chi^{(3)}\|_{L^\infty(\Omega)} \|\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}\|_{\mathbf{L}^\infty(\Omega)} \|\mathbf{r}_{\mathbf{E}}^n\|_{\varepsilon_0(1+\chi^{(1)})} \|\mathbf{r}_{\mathbf{E}}^n\|_{\mathbf{L}^\infty(\Omega)} \\
&\cdot \|\eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})} \\
&\le (\Delta t)^2 \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\right] \\
&\quad \times \|\mathbf{r}_{\mathbf{E}}^n\|_{\mathbf{L}^\infty(\Omega)} \|\mathbf{r}_{\mathbf{E}}^n\|_{\varepsilon_0(1+\chi^{(1)})} \|\eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})}.
\end{aligned}$$

Hence it remains to observe that

$$
\begin{aligned}
\|\mathbf{r}_{\mathbf{E}}^n\|_{\mathbf{L}^\infty(\Omega)} &\le \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))} \quad \text{(by Lemma 3.10(1)) and} \\
\|\mathbf{r}_{\mathbf{E}}^n\|_{\varepsilon_0(1+\chi^{(1)})}\|\eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})} &\le \frac{1}{2}\Delta t\, \|\mathbf{r}_{\mathbf{E}}^n\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{1}{2\Delta t}\,\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} \\
&\quad \text{(by Lemma 5.1(2) with } \alpha := \Delta t) \\
&\le \frac{1}{2}\|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} + \frac{1}{2\Delta t}\,\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} \\
&\quad \text{(by Lemma 3.10(1)).}
\end{aligned}
$$

So we get

$$
\begin{aligned}
\tilde{\delta}_{93}^n \le{}& \frac{1}{2}(\Delta t)^2 \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\right] \\
&\times \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))}\|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} \\
&+ \frac{1}{2}\|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\right] \\
&\times \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))}\Delta t\,\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.
\end{aligned}
$$

The term

$$
\tilde{\delta}_{94}^n = \Delta t \left(\varepsilon_0 \chi^{(3)}[\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T \eta^{n-1}\mathbf{r}_{\mathbf{E}}^n, \eta_h^n\right)
$$

can be estimated as $\tilde{\delta}_{91}^n$ (with η^n replaced by η^{n-1}), thus

$$
\begin{aligned}
\tilde{\delta}_{94}^n \le{}& C\Delta t\, h^{2k}\, \varepsilon_0 \|1+\chi^{(1)}\|_{L^\infty(\Omega)}\|\chi^{(3)}\|_{L^\infty(\Omega)}\Big[\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{L}^\infty(\Omega))} \\
&+ \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Big]\|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))}\|\mathbf{E}\|^2_{\mathbf{C}(0,T,\mathbf{H}^k(\Omega))} \\
&+ \frac{\Delta t}{2}\,\|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{L}^\infty(\Omega))} + \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\right] \\
&\times \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))}\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.
\end{aligned}
$$

Similarly

$$
\tilde{\delta}_{95}^n = -\Delta t \left(\varepsilon_0 \chi^{(3)}[\mathbf{E}(t^n) + \mathbf{E}_h^{n-1}]^T \eta_h^{n-1}\mathbf{r}_{\mathbf{E}}^n, \eta_h^n\right)
$$

is estimated as $\tilde{\delta}_{92}^n$ (with one if the terms η_h^n replaced by η_h^{n-1}):

$$
\begin{aligned}
\tilde{\delta}_{95}^n \le{}& \Delta t\, \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}(t^n)\|_{\mathbf{L}^\infty(\Omega)} + \|\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega)}\right] \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))} \\
&\times \|\eta_h^{n-1}\|_{\varepsilon_0(1+\chi^{(1)})}\|\eta_h^n\|_{\varepsilon_0(1+\chi^{(1)})} \\
\le{}& \frac{1}{2}\Delta t\, \|\chi^{(3)}\|_{L^\infty(\Omega)} \left[\|\mathbf{E}(t^n)\|_{\mathbf{L}^\infty(\Omega)} + \|\mathbf{E}_h^n\|_{\mathbf{L}^\infty(\Omega)}\right] \|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))} \\
&\times \left[\|\eta_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\right].
\end{aligned}
$$

Summarizing the estimates of $\tilde{\delta}_{91}^n$ to $\tilde{\delta}_{95}^n$, we conclude that

$$\begin{aligned}\tilde{\delta}_9^n &\le C_1 \Delta t\, h^{2k} + C_2 (\Delta t)^2 \|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(t^{n-1},t^n,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} \\ &\quad + C_3 \Delta t \left[\|\eta_h^{n-1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} \right],\end{aligned}$$

where the constant $C_1 > 0$ depends on ε_0, $\|1+\chi^{(1)}\|_{L^\infty(\Omega)}$, $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{H}^k(\Omega))}$, the constant $C_2 > 0$ depends on $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))}$, and the constant $C_3 > 0$ depends on $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))}$. It follows that

$$\begin{aligned}\tilde{\delta}_9 = \sum_{n=1}^N \tilde{\delta}_9^n &\le C_1 T h^{2k} + C_2 (\Delta t)^2 \|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} \\ &\quad + C_3 \Delta t\, \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2C_3 \Delta t\, S_\eta^N. \end{aligned} \tag{3.97}$$

The term

$$\begin{aligned}\tilde{\delta}_{10}^n &= 2\big(\varepsilon_0 \chi^{(3)} \big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big)\big] \\ &\quad \cdot \big(\mathbf{E}(t^n) - \mathbf{E}(t^{n-1})\big), \eta_h^n\big) \\ &= 2\Delta t \left(\varepsilon_0 \chi^{(3)} \big[2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big)\big]\, \mathbf{r}_{\mathbf{E}}^n, \eta_h^n\right)\end{aligned}$$

does not allow such a symmetric estimation argument as $\tilde{\delta}_9^n$. Here we start with

$$\begin{aligned}&2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big) \\ &= \mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T \\ &= \mathbf{E}(t^n)[\mathbf{E}(t^n) - \mathbf{E}_h^n]^T + [\mathbf{E}(t^n) - \mathbf{E}_h^n]\,[\mathbf{E}_h^n]^T \\ &\quad + \mathbf{E}(t^n)[\mathbf{E}(t^n) - \mathbf{E}_h^{n-1}]^T + [\mathbf{E}(t^n) - \mathbf{E}_h^{n-1}]\left[\mathbf{E}_h^{n-1}\right]^T.\end{aligned}$$

From

$$\begin{aligned}\mathbf{E}(t^n) - \mathbf{E}_h^n &= \eta^n - \eta_h^n, \\ \mathbf{E}(t^n) - \mathbf{E}_h^{n-1} &= \mathbf{E}(t^n) - \mathbf{E}(t^{n-1}) + \mathbf{E}(t^{n-1}) - \mathbf{E}_h^{n-1} \\ &= \Delta t\, \mathbf{r}_{\mathbf{E}}^n + \eta^{n-1} - \eta_h^{n-1}\end{aligned}$$

we obtain:

$$\begin{aligned}&2\mathbf{E}(t^n)[\mathbf{E}(t^n)]^T - \big(\mathbf{E}_h^n\,[\mathbf{E}_h^n]^T + \mathbf{E}_h^{n-1}\left[\mathbf{E}_h^{n-1}\right]^T\big) \\ &= \mathbf{E}(t^n)[\eta^n - \eta_h^n]^T + [\eta^n - \eta_h^n]\,[\mathbf{E}_h^n]^T \\ &\quad + \Delta t\, \mathbf{E}(t^n)[\mathbf{r}_{\mathbf{E}}^n]^T + \mathbf{E}(t^n)[\eta^{n-1} - \eta_h^{n-1}]^T \\ &\quad + \Delta t\, [\mathbf{r}_{\mathbf{E}}^n]\left[\mathbf{E}_h^{n-1}\right]^T + [\eta^{n-1} - \eta_h^{n-1}]\left[\mathbf{E}_h^{n-1}\right]^T.\end{aligned}$$

This decomposition generates a decomposition of $\tilde{\delta}_{10}^n$ into ten terms in a natural way:

$$\tilde{\delta}_{10}^n = \sum_{j=1}^{10} \tilde{\delta}_{10j}^n,$$

where

$$\begin{aligned}
\tilde{\delta}_{101}^n &:= 2\Delta t \left(\varepsilon_0 \chi^{(3)} \mathbf{E}(t^n)[\eta^n]^T \, \mathbf{r}_{\mathbf{E}}^n, \eta_h^n\right) \\
&\vdots \\
\tilde{\delta}_{1010}^n &:= -2\Delta t \left(\varepsilon_0 \chi^{(3)} \eta_h^{n-1} \left[\mathbf{E}_h^{n-1}\right]^T \, \mathbf{r}_{\mathbf{E}}^n, \eta_h^n\right).
\end{aligned}$$

All these terms can be estimated similar to the terms $\tilde{\delta}_{9j}^n$ so that we get an analogous estimate:

$$\begin{aligned}
\tilde{\delta}_{10} \leq\ & 2C_1 T h^{2k} + 2C_2 (\Delta t)^2 \|\partial_t \mathbf{E}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))} \\
& + 2C_3 \Delta t \, \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + 4C_3 \Delta t \, S_\eta^N. \qquad (3.98)
\end{aligned}$$

Finally we have to deal with the terms $\tilde{\delta}_{11}^n$ and $\tilde{\delta}_{12}^n$. Due to (3.84) it holds that

$$\begin{aligned}
\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}} &= \frac{1}{2}\big((\mathbf{E}_h^n)^2 + (\mathbf{E}_h^{n-1})^2\big) - \frac{1}{2}\big((\mathbf{E}_h^{n-1})^2 - (\mathbf{E}_h^{n-2})^2\big) \\
&= \frac{1}{2}\big((\mathbf{E}_h^n)^2 - (\mathbf{E}_h^{n-2})^2\big) \\
&= \frac{1}{2}\big(\mathbf{E}_h^n + \mathbf{E}_h^{n-2}\big)\big(\mathbf{E}_h^n - \mathbf{E}_h^{n-2}\big) \\
&= \frac{1}{2}\Delta t \big(\mathbf{E}_h^n + \mathbf{E}_h^{n-2}\big) \left(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\Delta t} + \frac{\mathbf{E}_h^{n-1} - \mathbf{E}_h^{n-2}}{\Delta t}\right).
\end{aligned}$$

By means of the discrete time derivative (3.76) this relation can be written as

$$\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}} = \frac{1}{2}\Delta t \big(\mathbf{E}_h^n + \mathbf{E}_h^{n-2}\big)\big(\partial_{\Delta t}\mathbf{E}_h^n + \partial_{\Delta t}\mathbf{E}_h^{n-1}\big),$$

and it follows that

$$\begin{aligned}
\|\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}}\|_{\mathbf{L}^\infty(\Omega)} &\leq \frac{1}{2}\Delta t \, \|\mathbf{E}_h^n + \mathbf{E}_h^{n-2}\|_{\mathbf{L}^\infty(\Omega)} \|\partial_{\Delta t}\mathbf{E}_h^n + \partial_{\Delta t}\mathbf{E}_h^{n-1}\|_{\mathbf{L}^\infty(\Omega)} \\
&\leq 2\Delta t \, \|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))} \|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}.
\end{aligned}$$

Thus we get

$$\begin{aligned}\tilde{\delta}_{11}^n &= \big(\varepsilon_0\chi^{(3)}\big[\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}}\big]\eta_h^n, \eta_h^n\big)\\ &\le \|\chi^{(3)}\|_{L^\infty(\Omega)}\|\mathbf{C}_1^{n+\frac{1}{2}} - \mathbf{C}_1^{n-\frac{1}{2}}\|_{\mathbf{L}^\infty(\Omega)}\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}\\ &\le 2\Delta t\,\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.\end{aligned}$$

The summation over n from 1 to $N-1$ gives

$$\tilde{\delta}_{11} = \sum_{n=1}^{N-1}\tilde{\delta}_{11}^n \le 2\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Delta t\, S_\eta^N. \tag{3.99}$$

The estimate of

$$\tilde{\delta}_{12}^n = \big(\varepsilon_0\chi^{(3)}\big[\mathbf{C}_2^{n+\frac{1}{2}} - \mathbf{C}_2^{n-\frac{1}{2}}\big]\eta_h^n, \eta_h^n\big)$$

runs in the same way. By (3.85) we have that

$$\mathbf{C}_2^{n+\frac{1}{2}} - \mathbf{C}_2^{n-\frac{1}{2}} = \Delta t\left[\mathbf{E}_h^n + \mathbf{E}_h^{n-2}\right]\Big([\partial_{\Delta t}\mathbf{E}_h^n]^T + \big[\partial_{\Delta t}\mathbf{E}_h^{n-1}\big]^T\Big),$$

and from this it follows that

$$\tilde{\delta}_{12}^n \le 4\Delta t\,\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}.$$

So we get

$$\tilde{\delta}_{12} = \sum_{n=1}^{N-1}\tilde{\delta}_{11}^n \le 4\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\Delta t\, S_\eta^N. \tag{3.100}$$

No we are ready to summarize the right-hand side of the inequality (3.87):

$$\begin{aligned}\sum_{j=1}^{12}\tilde{\delta}_j \le 3C_3\Delta t\,\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + C_4h^{2k} + C_5(\Delta t)^2\\ + C_6\Delta t\,S_\eta^N + 2\Delta t\,S_\theta^N,\end{aligned}$$

where the constant $C_4 > 0$ depends on T, ε_0, $\|1+\chi^{(1)}\|_{L^\infty(\Omega)}$, $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\partial_t\mathbf{E}\|_{\mathbf{L}^2(0,T,\mathbf{H}^k(\Omega))}$, $\|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{H}^k(\Omega))}$, $\|\partial_t\mathbf{H}\|_{\mathbf{L}^2(0,T,\mathbf{H}^{k+1}(\Omega))}$, the constant $C_5 > 0$ depends on $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\partial_t\mathbf{E}\|_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))}$, $\|\partial_{tt}\mathbf{E}\|_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\varepsilon_0(1+\chi^{(1)})}(\Omega))}$,

$\|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\partial_{tt}\mathbf{H}\|_{\mathbf{L}^2(0,T,\mathbf{L}^2_{\mu_0}(\Omega))}$, the constant $C_6 > 0$ depends on $\|\chi^{(3)}\|_{L^\infty(\Omega)}$, $\|(\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|(\partial_{\Delta t}\mathbf{E}_h^n)\|_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$, $\|\mathbf{E}\|_{\mathbf{C}^1(0,T,\mathbf{L}^\infty(\Omega))}$.

So we get from the inequality (3.87):

$$\begin{aligned}
&\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{N-\frac{1}{2}}\eta_h^N,\eta_h^N\right) + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{N-\frac{1}{2}}\eta_h^N,\eta_h^N\right) + \|\theta_h^N\|^2_{\mu_0}\\
&\le 3C_3\Delta t\,\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_1^{\frac{1}{2}}\eta_h^0,\eta_h^0\right)\\
&\quad + \left(\varepsilon_0\chi^{(3)}\mathbf{C}_2^{\frac{1}{2}}\eta_h^0,\eta_h^0\right) + \|\theta_h^0\|^2_{\mu_0} + C_4h^{2k} + C_5(\Delta t)^2 + C_6\Delta t\,S_\eta^N + 2\Delta t\,S_\theta^N.
\end{aligned} \tag{3.101}$$

Making use of the facts that

$$\begin{aligned}
&\left(\varepsilon_0\chi^{(3)}\mathbf{C}_j^{N-\frac{1}{2}}\eta_h^N,\eta_h^N\right) \ge 0,\\
&\qquad\left(\varepsilon_0\chi^{(3)}\mathbf{C}_j^{\frac{1}{2}}\eta_h^0,\eta_h^0\right) \le j\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})}, \quad j=1,2,
\end{aligned}$$

(cf. the estimates of $\tilde\delta_{11}^n$ and $\tilde\delta_{12}^n$), we finally conclude from (3.101) that

$$\begin{aligned}
&\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|^2_{\mu_0}\\
&\le 3C_3\Delta t\,\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})}\\
&\quad + 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^0\|^2_{\mu_0}\\
&\quad + C_4h^{2k} + C_5(\Delta t)^2 + C_6\Delta t\,S_\eta^N + 2\Delta t\,S_\theta^N\\
&\le C_7\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^0\|^2_{\mu_0}\\
&\quad + C_4h^{2k} + C_5(\Delta t)^2 + C_6\Delta t\,S_\eta^N + 2\Delta t\,S_\theta^N,
\end{aligned} \tag{3.102}$$

where $C_7 := 3C_3 + 1 + 3\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(\mathbf{E}_h^n)\|^2_{\ell^\infty(0,T,\mathbf{L}^\infty(\Omega))}$. Here we have used that Δt can be bounded by 1, for instance, without loss of generality.

It remains to apply Gronwall's inequality (Lemma 5.3) with
$\delta := \Delta t \ge 0$,
$g_0 := C_7\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^0\|^2_{\mu_0} + C_4h^{2k} + C_5(\Delta t)^2 \ge 0$,
$a_n := \left[\|\eta_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^n\|^2_{\mu_0}\right] \ge 0$,
$b_n := c_n := 0$,
$\gamma_0 := 0$, $\gamma_n := \gamma := \max\{C_6; 2\} \ge 0$ for $n \in \mathbb{N}$.

Then the condition $\gamma\delta < 1$ gives some (uniform) restriction of Δt. If we even require that $\Delta t < (2\max\{C_6;2\})^{-1}$, then we get by Lemma 5.3 that

$$\begin{aligned}
&\|\eta_h^N\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|^2_{\mu_0}\\
&\le \left[C_7\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^0\|^2_{\mu_0} + C_4h^{2k} + C_5(\Delta t)^2\right]\exp\left(\gamma\Delta t\sum_{n=1}^N(1-\gamma\Delta t)^{-1}\right)\\
&\le \left[C_7\|\eta_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^0\|^2_{\mu_0} + C_4h^{2k} + C_5(\Delta t)^2\right]\exp\left(2\gamma T\right).
\end{aligned}$$

If we take $\mathbf{E}_h^0 := \tilde{\mathbf{P}}_h \mathbf{E}(0) = \tilde{\mathbf{P}}_h \mathbf{E}_0$ and $\mathbf{H}_h^0 := \Pi_{1h}\mathbf{H}(0) = \Pi_{1h}\mathbf{H}_0$, we obtain

$$\|\eta_h^N\|_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_h^N\|_{\mu_0} \leq C\left[h^k + \Delta t\right] \exp\left(\gamma T\right),$$

where the constant $C > 0$ involves all the dependencies of the above constants C_1 to C_7. Finally, by the triangle inequality, we see that

$$\begin{aligned}
&\|\mathbf{E}(t^N) - \mathbf{E}_h^N\|_{\varepsilon_0(1+\chi^{(1)})} + \|\mathbf{H}(t^N) - \mathbf{H}_h^N\|_{\mu_0} \\
&\leq \|(\mathbf{I} - \tilde{\mathbf{P}}_h)\mathbf{E}(t^N)\|_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_h^N\|_{\varepsilon_0(1+\chi^{(1)})} + \|(\mathbf{I} - \Pi_{1h})\mathbf{H}(t^N)\|_{\mu_0} + \|\theta_h^N\|_{\mu_0},
\end{aligned}$$

so the estimates(1.10) and (1.13) imply that

$$\begin{aligned}
&\|\mathbf{E}(t^N) - \mathbf{E}_h^N\|_{\varepsilon_0(1+\chi^{(1)})} + \|\mathbf{H}(t^N) - \mathbf{H}_h^N\|_{\mu_0} \\
&\leq C\sqrt{\varepsilon_0}\, h^k \|\mathbf{E}\|_{\mathbf{C}(0,T,\mathbf{H}^k(\Omega))} + C\sqrt{\mu_0}\, h^k \|\mathbf{H}\|_{\mathbf{C}(0,T,\mathbf{H}^{k+1}(\Omega))} \\
&\quad + C\left[h^k + \Delta t\right] \exp\left(\gamma T\right).
\end{aligned}$$

$\square$

This theorem shows that the fully discrete (backward Euler-type) method for the nonlinear Maxwell's equations is unconditionally stable in the sense that there is no restriction to the relation between time step size and spatial grid size.

3.6 Numerical Results, Validations and Discussion

The full discretization of the system of nonlinear partial differential equations (3.6)–(3.8) leads to the nonlinear system of difference equations (3.58)–(3.60), which is solved by means of Newton's method. Thanks to the special structure of the time-discretized nonlinearity, at each time step the Newton iterations reduce to a single Euler-like backward step, making the whole procedure competitive.

Example 3.12 The permittivity, conductivity and the permeability are chosen as $\varepsilon = 1.0$, $\sigma = 0.0$ and $\mu = 2.0$. The susceptibilities $\chi^{(1)}$ and $\chi^{(3)}$ also assume constant values, but may be different in different tests. The electric and magnetic fields are initialized by taking the projections (1.15)–(1.18) of the exact electric and magnetic fields, where the exact fields given by [98]

$$\mathbf{E} = \Big(-2t - 2x, -2t - 2y, -2t - 2z \Big)^T, \quad \mathbf{H} = \Big(y - z, z - x, x - y \Big)^T,$$
$$\mathbf{J} = \Big(t + x, t + y, t + z \Big)^T.$$

Example 3.13 This test example is characterized by the following parameters. The permittivity and the permeability are chosen as the constant vacuum values $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$. The susceptibilities $\chi^{(1)}$ and $\chi^{(3)}$ also assume constant values, but may be different in different tests. The angular frequency is $\omega = 2\pi f$ (rad$\cdot$s^{-1}) with $f = \frac{\sqrt{3}}{2} c_0$ Hz. The exact electric and magnetic fields are given as in [12]:

$$\begin{aligned}
\mathbf{E}_1(t) &= -\cos(\pi x)\sin(\pi y)\sin(\pi z)\cos(\omega t),\\
\mathbf{E}_2(t) &= 0,\\
\mathbf{E}_3(t) &= \sin(\pi x)\sin(\pi y)\cos(\pi z)\cos(\omega t),\\
\mathbf{H}_1(t) &= -\frac{\pi}{\omega}\sin(\pi x)\cos(\pi y)\cos(\pi z)\sin(\omega t),\\
\mathbf{H}_2(t) &= \frac{2\pi}{\omega}\cos(\pi x)\sin(\pi y)\cos(\pi z)\sin(\omega t),\\
\mathbf{H}_3(t) &= -\frac{\pi}{\omega}\cos(\pi x)\cos(\pi y)\sin(\pi z)\sin(\omega t).
\end{aligned}$$

When $\chi^{(3)} = 0$ in the system of equations (3.1)–(3.5), the problem becomes linear. For this case, error estimates both at semi-discrete and fully discrete levels, energy conservation and simulations have already been demonstrated in [8, 9, 10, 12]. Here, the accuracy and energy stability for the nonlinear 3D problem, at semi-discrete and fully discrete levels, are theoretically founded by several theorems and computational experiments. Theorem 3.3 states that the nonlinear problem (3.1)–(3.5) in 3D is well-posed and conserves the energy at the continuous level. The Theorem 3.5 demonstrates that the nonlinear effects in the electric polarization at the semi-discrete level also conserves the energy. Similarly the Theorem 3.8 says that the nonlinear electromagnetic energy is stable at the fully discrete level. The Theorems 3.7 and 3.11 demonstrate semi-discrete and fully discrete a priori estimates of the absolute error, and these results are optimal within the selected finite element spaces. Our proposed methods solve numerically the full system of Maxwell's equations with cubic nonlinearities in 3D directly, whereas many existing methods do not solve the nonlinear problem directly [54, 62, 25, 125, 74, 116, 109, 79, 128, 42, 15] or solve the problem only in 1D and 2D, e.g. [48],[133, 132, 131, 141, 95, 134, 22, 68, 3, 4].

The electric field and magnetic induction are visualized for various 3D meshes (beam , Fichera and Escher). In Figs. 3.1–3.4, the electric field and magnetic induction are initialized by taking the projections (1.15)–(1.18) of the exact quantities from Example 3.12. The snapshots of the electric field and magnetic induction in Figs. 3.1–3.4 present the results obtained using the backward Euler-type method at the time $T = 10^{-7}$, where the time step size is $\Delta t = 10^{-9}$, and the susceptibilities parameters are $\chi^{(1)} = 3.2$ and $\chi^{(3)} = 1.2$. Fig. 3.1 shows the electric field and magnetic induction values for the beam mesh at the final time. In Fig 3.2, different orientations of the electric field (Fichera mesh 3D L-shaped domain) at the final time $T = 10^{-7}$ are depicted. Similarly the magnetic induction is presented in Fig. 3.3 at the final time.

Snapshots of the electric field and magnetic induction taken at the final time $T = 10^{-7}$ are presented in Fig. 3.4 for the Escher mesh.

In Figs. 3.5–3.6, the projections (1.15)–(1.18) of the exact quantities from Example 3.13 are used to initialize the electric field and magnetic induction. Figs. 3.5 and 3.6 depict the electric field and magnetic induction for a Fichera mesh at the time $T = 0.001$, where the time step size is $\Delta t = 0.00001$, and the susceptibilities parameters are $\chi^{(1)} = 2.2$ and $\chi^{(3)} = 1.5 \times 10^{-19}$.

A number of numerical experiments was performed to validate the energy conserving properties of the proposed methods, by employing the backward Euler-type and SDIRK23 [59, 1] methods. In Figs. 3.7–3.16, the permittivity, permeability, susceptibilities $\chi^{(1)}$, and $\chi^{(3)}$ also assume constant values to determine the energy (3.61).

In Figs. 3.7, 3.8, the energy (3.61) from $t_0 = 0.0001$ to the final time $T = 0.001$ is presented by employing the backward Euler-type and SDIRK23 methods. The time step size is $\Delta t = 0.0001$. Furthermore, in Figs. 3.9, 3.10

the energy (3.61) for the time step size $\Delta t = 0.00001$, from $t_0 = 0.00001$ to the final time $T = 0.0003$, is depicted by employing the backward Euler-type and SDIRK23 methods. The parameters are $\chi^{(1)} = 3.2222$ and $\chi^{(3)} = 1.5 \times 10^{-19}$ in Figs. 3.7–3.10.

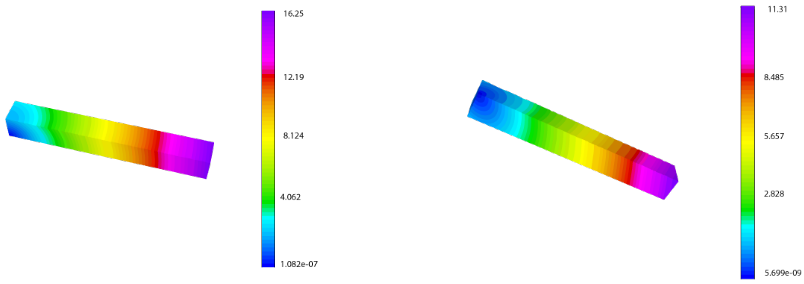
In Figs. 3.11, 3.12, the susceptibilities $\chi^{(1)} = 2.2$ and $\chi^{(3)} = 4.1$ are chosen, where the time set size is $\Delta t = 0.00001$. The Figs. 3.11, 3.12 show the energy (3.61) from $t_0 = 0.00001$ to the final time $T = 0.0003$ by using the backward Euler-type and SDIRK23 methods, respectively.

The energy (3.61) obtained by the backward Euler-type and SDIRK23 methods from $t_0 = 0.000001$ to the final time $T = 0.00003$ is presented in Figs. 3.13, 3.14; the parameters are $\Delta t = 0.000001$, $\chi^{(1)} = 3.2222$ and $\chi^{(3)} = 1.5 \times 10^{-19}$.

The Figs. 3.15, 3.16 demonstrate the energy (3.61) from $t_0 = 10^{-9}$ to the final time $T = 3 \times 10^{-8}$ obtained by using the backward Euler-type and SDIRK23 methods with the time step size $\Delta t = 10^{-9}$, where the susceptibility parameters are chosen as $\chi^{(1)} = 3.2222$, $\chi^{(3)} = 1.5 \times 10^{-19}$ (Fig. 3.15) and $\chi^{(1)} = 2.2$, $\chi^{(3)} = 4.1$ (Fig. 3.16).

The Figs. 3.7–3.16 illustrate the conservation property of the energy (3.61) for the nonlinear problem in 3D. We showed that the semi discretization (3.29)–(3.31) along with SDIRK23 [1, 59] method also conserves the energy. The proposed time discretizations (backward Euler-type and SDIRK23 methods) of the nonlinear problem in 3D are unconditionally stable, but the SDIRK23 method is computationally more expensive in contrast to the backward Euler-type method.

We conclude from the theorems and computational experiments presented in this chapter that the proposed novel TDFEMs for the full system of nonlinear Maxwell's equation in 3D conserve the energy (at semi-discrete and fully discrete levels), have optimal solutions, and figure out the fields (quantities) directly, in contrast to the electric field formulation, magnetic field formulation, A-formulation [134], operator-form and the decoupled scheme, for details see [12]. Moreover our proposed methods are intermediate results for the theoretical and computational development of energy conserving time-domain discontinuous methods for 3D nonlinear problems in optics and photonics. In particular, our proposed semi-and fully discrete methods could replace the existing 1D [22] and 2D [3] schemes to 3D, and A-formulation [134].

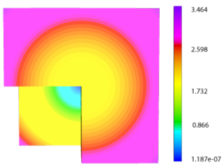


(a) electric field (b) magnetic induction

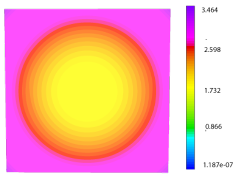
Figure 3.1: A snapshot of the electric field (3.1a) and magnetic induction (3.1b) are taken at the final time $T = 10^{-7}$ (number of steps $N = 100$) using the backward Euler-type method for a beam mesh. The parameters are: time step size $\Delta t = 10^{-9}$, $\chi^{(1)} = 2.2$ and $\chi^{(3)} = 4.1$.

3.7 Summary

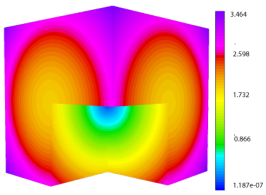
In this chapter, a new modeling approach has been developed that allows the direct time integration of the full nonlinear Maxwell's equations in optics and photonics in 3D. The new capabilities of the proposed method permit that linear and nonlinear effects of the electric polarization in 3D are modeled in an efficient manner that is unconditionally stable and conserves the energy. The novel approach allows energy stability and error estimates both at the semi-discrete and fully discrete levels, which were not yet available using the discontinous spaces or edge or face elements with the Euler time discretization for the full system of nonlinear Maxwell's equations in 3D. The approach is almost completely general and could replace the electric filed formulation, magnetic filed formulation, and A-Formulation. Numerical results of energy validate the theoretical findings, which prove that the full discretization is unconditionally stable and conserves the energy.



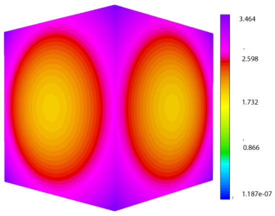
(a) Snapshot from above



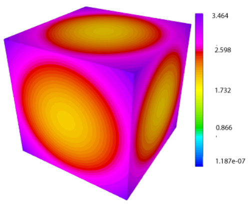
(b) Snapshot from below



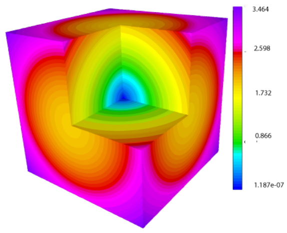
(c) Snapshot from above and front



(d) Snapshot from back and below

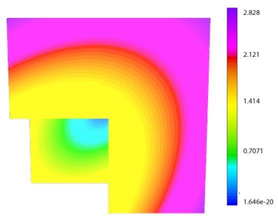


(e) Snapshot from back, left and below

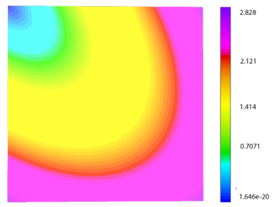


(f) Snapshot from right, front and above

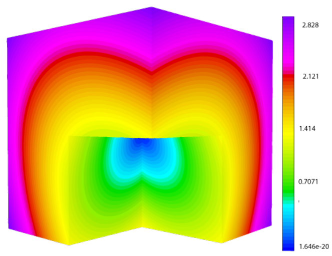
Figure 3.2: The snapshots of the electric field are taken at the final time $T = 10^{-7}$ (number of step $N = 100$) using the backward Euler-type method for the Fichera mesh. The parameters are: time step size $\Delta t = 10^{-9}$, $\chi^{(1)} = 2.2$ and $\chi^{(3)} = 4.1$.



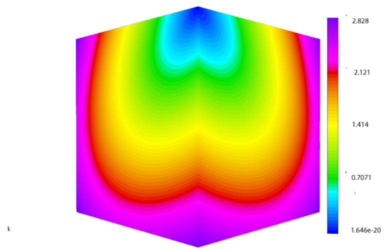
(a) Snapshot from above



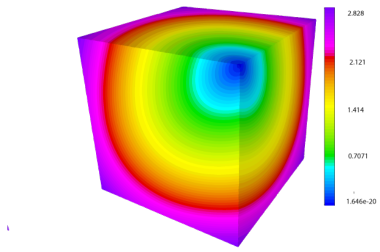
(b) Snapshot from below



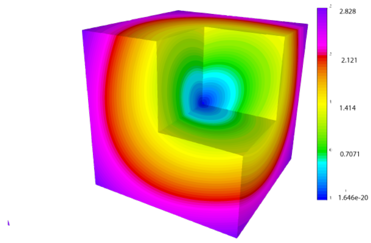
(c) Snapshot from above and front



(d) Snapshot from back and below

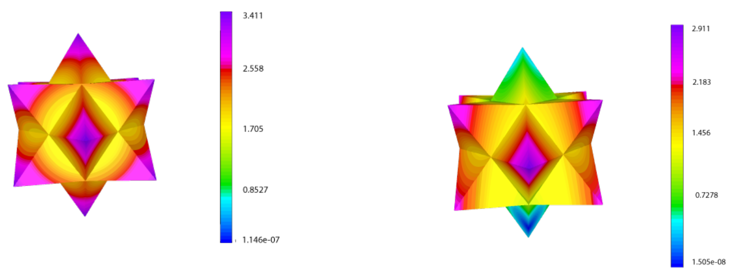


(e) Snapshot from back, left and below



(f) Snapshot from right, front and above

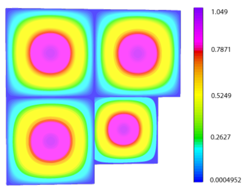
Figure 3.3: The snapshots of the magnetic induction are taken at the final time $T = 10^{-7}$ (number of step $N = 100$) using the backward Euler-type method for the Fichera mesh. The parameters are: time step size $\Delta t = 10^{-9}$, $\chi^{(1)} = 2.2$ and $\chi^{(3)} = 4.1$.



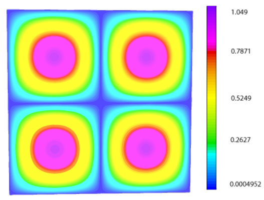
(a) electric field

(b) magnetic induction

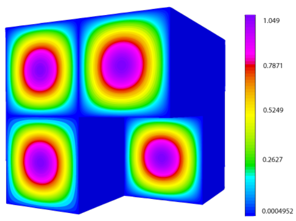
Figure 3.4: A snapshot of the electric field and magnetic induction at the final time $T = 10^{-7}$ (number of steps $N = 100$) using the backward Euler-type method for an Escher mesh is taken. The parameters are: time step size $\Delta t = 10^{-9}$, $\chi^{(1)} = 2.2$ and $\chi^{(3)} = 4.1$.



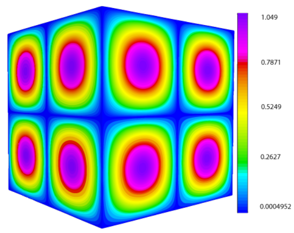
(a) Snapshot from above



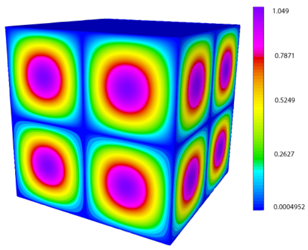
(b) Snapshot from below



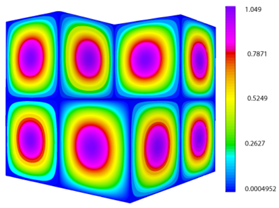
(c) Snapshot from above and front



(d) Snapshot from back and below

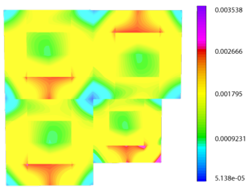


(e) Snapshot from back, left and below

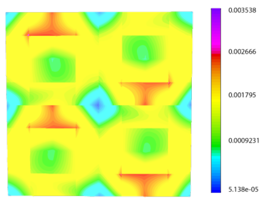


(f) Snapshot from right and front

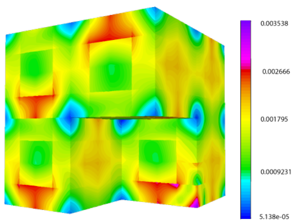
Figure 3.5: The snapshots of the electric field are taken at the final time $T = 0.001$ (number of steps $N = 100$) using the backward Euler-type method for the Fichera mesh. The parameters are: time step size $\Delta t = 0.00001$, $\chi^{(1)} = 2.2$ and $\chi^{(3)} = 1.5 \times 10^{-19}$.



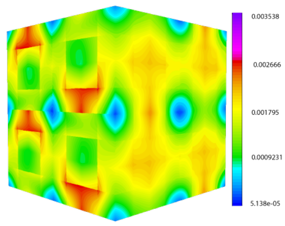
(a) Snapshot from above



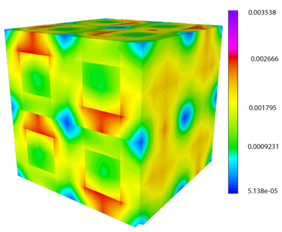
(b) Snapshot from below



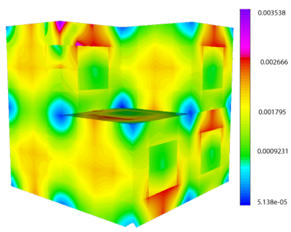
(c) Snapshot from above and front



(d) Snapshot from the back and below

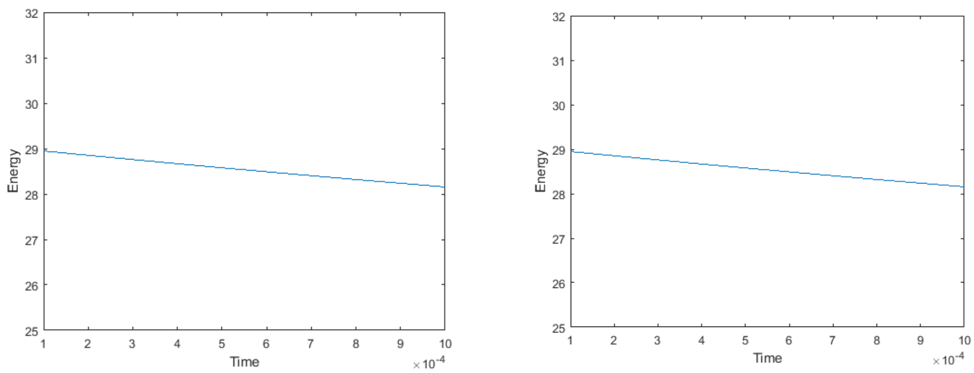


(e) Snapshot from the back, left and below



(f) Snapshot from right and front

Figure 3.6: The snapshots of the magnetic induction are taken at the final time $T = 0.001$ (number of steps $N = 100$) using the backward Euler-type method for the Fichera mesh. The parameters are: time step size $\Delta t = 0.00001$, $\chi^{(1)} = 2.2$ and $\chi^{(3)} = 1.5 \times 10^{-19}$.



(a) backward Euler-type (b) SDIRK23

Figure 3.7: The energy (3.61) from $t_0 = 0.0001$ to the final time $T = 0.001$ are presented by employing the backward Euler-type and SDIRK23 methods. The parameters are: time step size $\Delta t = 0.0001$, $\chi^{(1)} = 3.2222$ and $\chi^{(3)} = 1.5 \times 10^{-19}$.

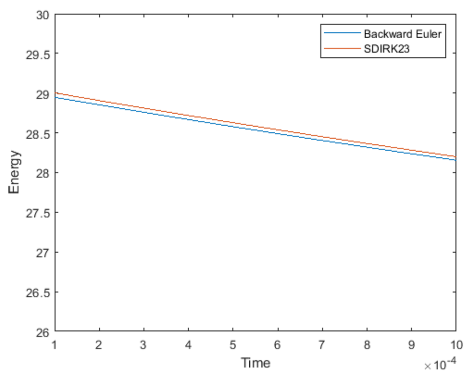
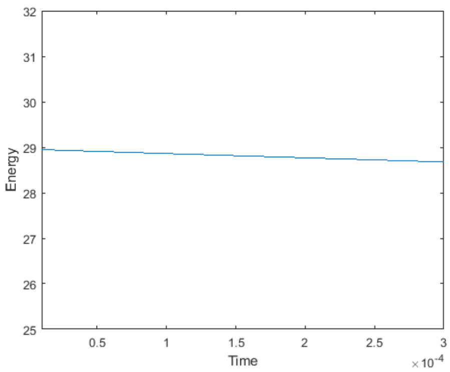
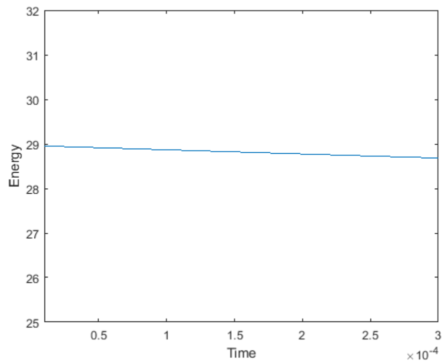


Figure 3.8: The energy (3.61) from $t_0 = 0.0001$ to the final time $T = 0.001$ are presented by employing the backward Euler-type and SDIRK23 methods. The parameters are: time step size $\Delta t = 0.0001$, $\chi^{(1)} = 3.2222$ and $\chi^{(3)} = 1.5 \times 10^{-19}$.



(a) backward Euler-type



(b) SDIRK23

Figure 3.9: The energy (3.61) from $t_0 = 0.00001$ to the final time $T = 0.0003$ are presented by employing the backward Euler-type and SDIRK23 methods. The parameters are: time step size $\Delta t = 0.00001$, $\chi^{(1)} = 3.2222$ and $\chi^{(3)} = 1.5 \times 10^{-19}$.

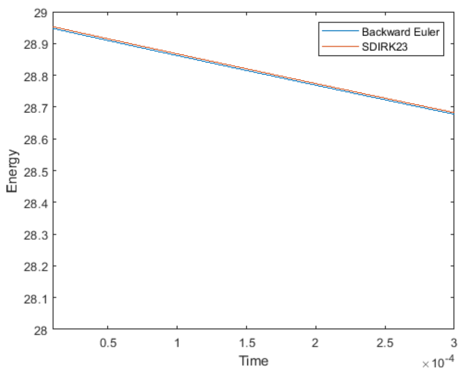
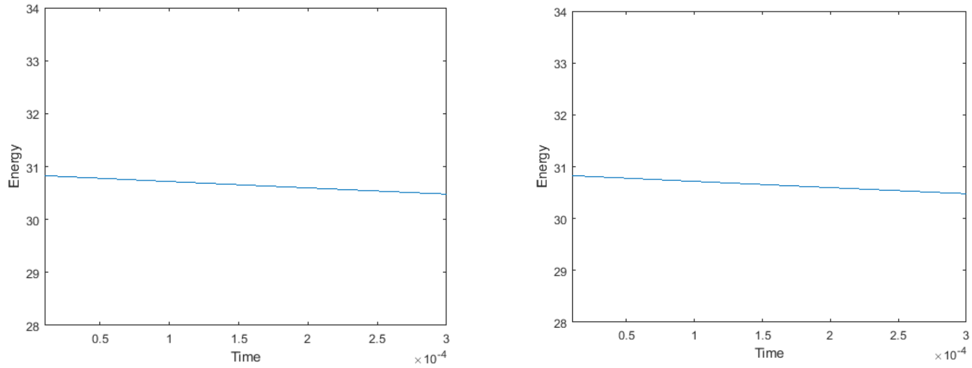


Figure 3.10: The energy (3.61) from $t_0 = 0.00001$ to the final time $T = 0.0003$ are presented by employing the backward Euler-type and SDIRK23 methods. The parameters are: time step size $\Delta t = 0.00001$, $\chi^{(1)} = 3.2222$ and $\chi^{(3)} = 1.5 \times 10^{-19}$.



(a) backward Euler-type (b) SDIRK23

Figure 3.11: The energy (3.61) from $t_0 = 0.00001$ to the final time $T = 0.0003$ are presented by employing the backward Euler-type and SDIRK23 methods. The parameters are: time step size $\Delta t = 0.00001$, $\chi^{(1)} = 2.2$ and $\chi^{(3)} = 4.1$.

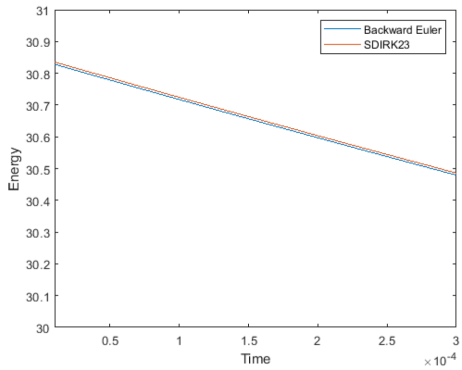
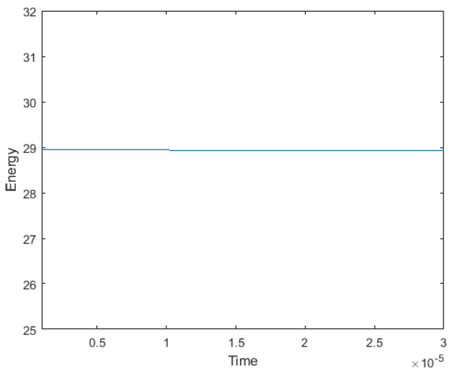
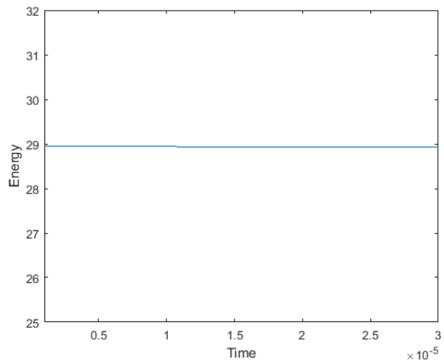


Figure 3.12: The energy (3.61) from $t_0 = 0.00001$ to the final time $T = 0.0003$ are presented by employing the backward Euler-type and SDIRK23 methods. The parameters are: time step size $\Delta t = 0.00001$, $\chi^{(1)} = 2.2$ and $\chi^{(3)} = 4.1$.



(a) backward Euler-type



(b) SDIRK23

Figure 3.13: The energy (3.61) from $t_0 = 0.000001$ to the final time $T = 0.00003$ are presented by employing the backward Euler-type and SDIRK23 methods. The parameters are: $\Delta t = 0.000001$, $\chi^{(1)} = 3.2222$ and $\chi^{(3)} = 1.5 \times 10^{-19}$.

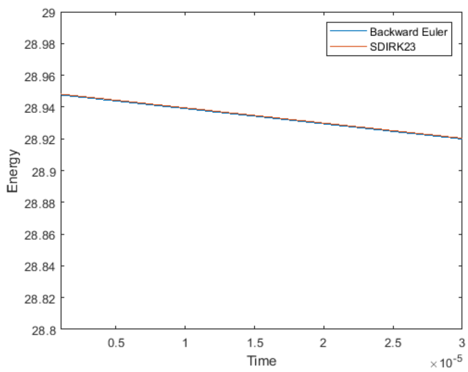


Figure 3.14: The energy (3.61) from $t_0 = 0.000001$ to the final time $T = 0.00003$ are presented by employing the backward Euler-type and SDIRK23 methods. The parameters are: $\Delta t = 0.000001$, $\chi^{(1)} = 3.2222$ and $\chi^{(3)} = 1.5 \times 10^{-19}$.

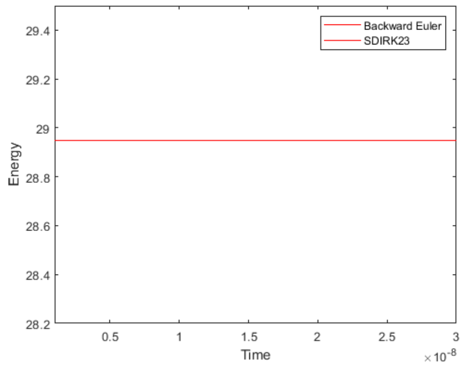


Figure 3.15: The energy (3.61) from $t_0 = 10^{-9}$ to the final time $T = 3 \times 10^{-8}$ are presented by employing the backward Euler-type and SDIRK23 methods. The parameters are: $\Delta t = 10^{-9}$, $\chi^{(1)} = 3.2222$ and $\chi^{(3)} = 1.5 \times 10^{-19}$.

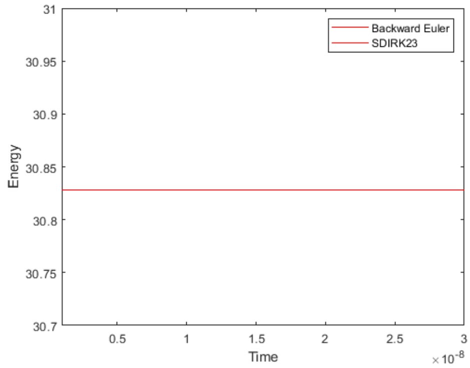


Figure 3.16: The energy (3.61) from $t_0 = 10^{-9}$ to the final time $T = 3 \times 10^{-8}$ are presented by employing the backward Euler-type and SDIRK23 methods. The parameters are: $\Delta t = 10^{-9}$, $\chi^{(1)} = 2.2$ and $\chi^{(3)} = 4.1$.

Chapter 4

Time Domain Discontinuous Galerkin Method for Optics and Photonics

In this chapter, we extend the semi-discrete mixed finite element method [8], [9], the fully discrete finite element method [10], [12] and the result of Chapter 3 to the time-dependent discontinuous Galerkin finite element method for Maxwell's equations with nonlinearities. Let $\Omega = (r, s) \times (p, q)$ be a rectangular domain in $\mathbb{R}^2$ with boundary Γ and unit outward normal $\mathbf{n}$. The nonlinear Maxwell's equations (3.1)–(3.2) can be rewritten as:

$$\partial_t \mathbf{D} - \nabla \times \mathbf{H} = \mathbf{J} \ \text{ in } \Omega \times (0, T), \tag{4.1}$$

$$\mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} = 0 \ \text{ in } \Omega \times (0, T), \tag{4.2}$$

where

$$\mathbf{D} = \varepsilon_0 \Big((1 + \chi^{(1)}) \mathbf{E} + \chi^{(3)} |\mathbf{E}|^2 \, \mathbf{E} \Big),$$

and its derivative

$$\partial_t \mathbf{D} = \varepsilon_0 \Big((1 + \chi^{(1)}) \partial_t \mathbf{E} + \chi^{(3)} \partial_t [|\mathbf{E}|^2 \, \mathbf{E}] \Big).$$

$$\partial_t \mathbf{D} = \varepsilon_0 \Big((1 + \chi^{(1)}) \partial_t \mathbf{E} + \chi^{(3)} [|\mathbf{E}|^2 + 2 \big(\mathbf{E}\mathbf{E}^T \big)] \partial_t \mathbf{E} \Big), \tag{4.3}$$

and

$$\mathbf{E}\mathbf{E}^T = \begin{pmatrix} E_x^2 & E_x E_y & E_x E_z \\ E_x E_y & E_y^2 & E_y E_z \\ E_x E_z & E_y E_z & E_z^2 \end{pmatrix}$$

A perfect conducting boundary condition on Ω is assumed so that

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on } \Gamma \times (0, T).$$

In addition, initial conditions have to be specified so that

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}) \quad \text{and} \quad \mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega,$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are given functions on Γ, and $\mathbf{H}_0$ satisfies

$$\nabla \cdot \mu_0 \mathbf{H}_0 = 0 \ \text{ in } \Omega, \ \mathbf{H}_0 \cdot \mathbf{n} = 0 \ \text{ on } \Gamma. \tag{4.4}$$

The divergence-free condition in (4.4) together with (4.2) implies that

$$\nabla \cdot \mu_0 \mathbf{H} = 0 \quad \text{in } \Omega \times (0, T). \tag{4.5}$$

The 2D transverse electric field mode in the z-direction is considered, which contains the electric displacement $\mathbf{D} := (D_x, D_y)$, electric field $\mathbf{E} := (E_x, E_y)$, $\nabla \times \mathbf{E} := (\partial_x E_y, -\partial_y E_x)$, magnetic field $\mathbf{H} := H_z$, $\nabla \times \mathbf{H} := (\partial_y H_z, -\partial_x H_z)^T$, current density $\mathbf{J} := (J_x, J_y)$. The subscripts x, y and z denothe the x-direction, y-direction, and z-direction respectively. In addition $\mathbf{E}^2 := E_x^2 + E_y^2$ and

$$\begin{aligned}
\mathbf{E}\mathbf{E}^T \partial_t \mathbf{E} &= \begin{pmatrix} E_x^2 & E_x E_y \\ E_x E_y & E_y^2 \end{pmatrix} \cdot \begin{pmatrix} \partial_t E_x \\ \partial_t E_y \end{pmatrix} \\
&= \begin{pmatrix} E_x^2 \partial_t E_x + E_x E_y \partial_t E_y \\ E_x E_y \partial_t E_x + E_y^2 \partial_t E_y \end{pmatrix}.
\end{aligned}$$

The equations (4.1)–(4.3) can be simplifed as

$$\partial_t D_x = \partial_y H_z + J_x, \tag{4.6}$$

$$\partial_t D_y = -\partial_x H_z + J_y, \tag{4.7}$$

$$\mu_0 \partial_t H_z = -\partial_x E_y + \partial_y E_x, \tag{4.8}$$

$$\begin{aligned}
\partial_t D_x = \varepsilon_0 \Big(&(1 + \chi^{(1)}) \partial_t E_x + \chi^{(3)} [\big(E_x^2 + E_y^2 \big) \partial_t E_x \\
&+ 2 \big(E_x^2 \partial_t E_x + E_x E_y \partial_t E_y \big)] \Big),
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\partial_t D_y = \varepsilon_0 \Big(&(1 + \chi^{(1)}) \partial_t E_y + \chi^{(3)} [\big(E_x^2 + E_y^2 \big) \partial_t E_y \\
&+ 2 \big(E_x E_y \partial_t E_x + E_y^2 \partial_t E_y \big)] \Big).
\end{aligned} \tag{4.10}$$

The initial conditions are defined as

$$E_x(x, 0) = E_x^0 \quad \text{and} \quad E_y(y, 0) = E_y^0 \quad \text{and} \qquad H_z(z, 0) = H_z^0.$$

The perfect conducting (PEC) boundary condition in 2D is

$$E_x(x, y, t)|_{y=p,q} = 0, \quad \text{and} \quad E_y(x, y, t)|_{x=r,s} = 0. \tag{4.11}$$

4.0.1 The Nonlinear Electromagnetic Energy at the Continuous Level

In this section, the nonlinear electromagnetic energy of the system (4.6)–(4.10) at any time t is defined by

$$\begin{aligned}\mathcal{E}(t) := &\Big[\|E_x\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_y\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big] + \|H_z\|^2_{\mu_0}\\ &+ \frac{3}{2}\Big\|E_x^2 + E_y^2\Big\|^2_{\varepsilon_0\chi^{(3)}}.\end{aligned}$$

In the next, we will prove that the nonlinear electromagnetic energy at any time t is conserved and bounded.

Theorem 4.1 *For a given $(J_x, J_y) = (0, 0)$ and (E_x, E_y, H_z) is the corresponding weak solution of the system (4.6)–(4.10), then the nonlinear electromagnetic energy of the system (4.6)–(4.10) at any time t satisfies*

$$\begin{aligned}&\Big[\|E_x\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_y\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big] + \|H_z\|^2_{\mu_0} + \frac{3}{2}\Big\|E_x^2 + E_y^2\Big\|^2_{\varepsilon_0\chi^{(3)}}\\ &= \Big[\|E_x^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_y^0\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big] + \|H_z^0\|^2_{\mu_0} + \frac{3}{2}\Big\|(E_x^0)^2 + (E_y^0)^2\Big\|^2_{\varepsilon_0\chi^{(3)}}.\end{aligned}$$

Proof: Multiplying the equations (4.6)–(4.10) by E_x, E_y, H_z, E_x, E_y respectively, integrating in space Ω and time $[0, T]$, adding the resulting equations after employing the boundary condition (4.11), complete the proof of Theorem (4.1). □

Remark 4.2 This result of the Theorem 4.1 can be obtained similar to the energy Theorem 4.3 at discrete levels.

The nonlinear problem (4.6)–(4.10) is solved in the rectangular domain Ω and is discretized by non-uniform grid

$$\begin{aligned}r := x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N_{x+\frac{1}{2}}} =: s,\\ p := y_{\frac{1}{2}} < y_{\frac{3}{2}} < \ldots < y_{N_{y+\frac{1}{2}}} =: q.\end{aligned}$$

The rectangular cells are defined as $K_{i,j} := I_i \times J_j$ and $I_i := [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $i = 0, 1, 2, 3, \ldots, N_x$. Similarly the cells $J_j := [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$, $j = 0, 1, 2, 3, \ldots, N_y$ are defined.

The mesh sizes are denoted by $h_i^x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and $h_j^y = y_{j+\frac{1}{2}} - y_{i-\frac{1}{2}}$ with $h_x^{max} = \max_{1\le i\le N_x} h_i^x$ and $h_y^{max} = \max_{1\le j\le N_y} h_j^y$. The maximal mesh size is defined by $h = \max\{h_x^{max}, h_y^{max}\}$. The mesh is rectangular and we assume that there exists a constant $C > 0$ independent of h such that $h_i^x \geq C \cdot h$ and $h_j^y \geq C \cdot h$, for all i, j. The finite element space U_h^k is the space of tensor products of piecewise polynomials of degree at most k in each variable on every element $K_{i,j}$ defined by

$$U_h^k := \{u : \ u|_K \in Q_k(K), \forall K \in \mathcal{T}_h\}.$$

The space $Q_k(K)$ contains tensor products of one dimensional polynomials of degree up to k, and the $\mathcal{T}_h$ denote the Cartesian grid on Ω with mesh size h. The symbol u_h is a numerical approximation of the corresponding variable u, and belongs to the space U_h^k. The functions in U_h^k are allowed to have discontinuities across the cell interfaces.

The limiting values of u_h at $x_{i+\frac{1}{2}}$ from the right cell of $K_{i+1,j}$ and left cell of the $K_{i.j}$ are denoted by $u_h(x^+_{i+\frac{1}{2}}, y)$ or $(u_h)^+_{i+\frac{1}{2},y}$, $u_h^+(x_{i+\frac{1}{2}}, y)$ and $u_h(x^-_{i+\frac{1}{2}}, y)$ or $(u_h)^-_{i+\frac{1}{2},y}$, $u_h^-(x_{i+\frac{1}{2}}, y)$ respectively. The numerical fluxes are obtained by integration by parts. The numerical fluxes should be considered and designed carefully to ensure conservation of energy, optimal error estimates and numerical stability. The numerical fluxes are the functions that are defined on the cell boundaries. The alternating fluxes are defined in a simple and elegant way like LDG (local discontinuous Galerkin) methods for diffusion equations, second order wave equation and Maxwell's equations [36, 130, 93]. c_0 is a constant that is independent of h. The alternative fluxes are

$$\hat{E}_{x,h}(x, y_{j+\frac{1}{2}}) := E^+_{x,h}(x, y_{j+\frac{1}{2}}) \quad \forall j = 1, 2, 3, \ldots, N_{y-1}, \tag{4.12}$$

$$\hat{E}_{x,h}(x, y_{\frac{1}{2}}) := E_{x,h}(x, y_{N_y+\frac{1}{2}}) = 0, \tag{4.13}$$

$$\hat{E}_{y,h}(x_{i+\frac{1}{2}}, y) := E^+_{y,h}(x_{i+\frac{1}{2}}, y) \quad \forall i = 1, 2, 3, \ldots, N_{x-1}, \tag{4.14}$$

$$\hat{E}_{y,h}(x_{\frac{1}{2}}, y) := E_{y,h}(x_{N_x+\frac{1}{2}}, y) = 0, \tag{4.15}$$

$$\hat{H}_{z,h}(x, y_{j+\frac{1}{2}}) := H^-_{z,h}(x, y_{j+\frac{1}{2}}) \quad \forall j = 1, 2, 3, \ldots, N_y, \tag{4.16}$$

$$\hat{H}_{z,h}(x, y_{\frac{1}{2}}) := H^+_{z,h}(x, y_{\frac{1}{2}}) + c_0[\![E_{x,h}(x, y_{\frac{1}{2}})]\!], \tag{4.17}$$

$$\hat{H}_{z,h}(x_{i+\frac{1}{2}}, y) := H^-_{z,h}(x_{i+\frac{1}{2}}, y) \quad \forall j = 1, 2, 3, \ldots, N_x, \tag{4.18}$$

$$\hat{H}_{z,h}(x_{\frac{1}{2}}, y) := H^+_{z,h}(x_{\frac{1}{2}}, y) - c_0[\![E_{y,h}(x_{\frac{1}{2}}, y)]\!], \tag{4.19}$$

and the jump is considered

$$[\![E_{x,h}(x, y_{\frac{1}{2}})]\!] := E^+_{x,h}(x, y_{\frac{1}{2}}) - 0, \tag{4.20}$$

$$[\![E_{y,h}(x_{\frac{1}{2}}, y)]\!] := E^+_{y,h}(x_{\frac{1}{2}}, y) - 0. \tag{4.21}$$

The standard notation for the jumps on cell boundaries throughout this chapter is defined by

$$[\![\Psi]\!] := \Psi^+ - \Psi^-.$$

For $c_0 = \frac{1}{2}$, the fluxes (4.17) and (4.19) match with the standard upwind fluxes

$$\hat{H}_{z,h}(x, y_{\frac{1}{2}}) := \frac{1}{2}[H^+_{z,h}(x, y_{\frac{1}{2}}) + H^-_{z,h}(x, y_{\frac{1}{2}})] + \frac{1}{2}[\![E_{x,h}(x, y_{\frac{1}{2}})]\!], \tag{4.22}$$

$$\hat{H}_{z,h}(x_{\frac{1}{2}}, y) := \frac{1}{2}[H^+_{z,h}(x_{\frac{1}{2}}, y) + H^-_{z,h}(x_{\frac{1}{2}}, y)] - \frac{1}{2}[\![E_{y,h}(x_{\frac{1}{2}}, y)]\!], \tag{4.23}$$

where the undefined $H^-_{z,h}(x, y_{\frac{1}{2}})$ and $H^-_{z,h}(x_{\frac{1}{2}}, y)$ are replaced by $H^+_{z,h}(x, y_{\frac{1}{2}})$ and $H^+_{z,h}(x_{\frac{1}{2}}, y)$, respectively.

4.1 Spatial Discretization for Discontinuous Galerkin Method

For the test functions Φ_{1h}, Φ_{2h} $\Phi_{3h} \in U_h^k$, the discontinuous Galerkin formulation for the equations (4.6)–(4.10) with respect to the semi-discrete solution $(E_{x,h}, E_{y,h}, H_{z,h}) \in C^1(0, T, U_h^k)$ reads as follow:

$$\begin{aligned}
&\int_{K_{i,j}} \partial_t D_{x,h}\Phi_{1,h} - \int_{I_i} [(\hat{H}_{z,h}\Phi^-_{1,h})_{x,j+\frac{1}{2}} - (\hat{H}_{z,h}\Phi^+_{1,h})_{x,j-\frac{1}{2}}]dx \\
&+ \int_{K_{i,j}} H_{z,h}\partial_y\Phi_{1,h} - \int_{K_{i,j}} J_{x,h}\Phi_{1,h} = 0,
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
&\int_{K_{i,j}} \partial_t D_{y,h}\Phi_{2,h} + \int_{J_j} [(\hat{H}_{z,h}\Phi^-_{2,h})_{i+\frac{1}{2},y} - (\hat{H}_{z,h}\Phi^+_{2,h})_{i-\frac{1}{2},y}]dy \\
&- \int_{K_{i,j}} H_{z,h}\partial_x\Phi_{2,h} - \int_{K_{i,j}} J_{y,h}\Phi_{2,h} = 0,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
&\mu_0 \int_{K_{i,j}} \partial_t H_{z,h}\Phi_{3,h} + \int_{J_j} [(\hat{E}_{y,h}\Phi^-_{3,h})_{i+\frac{1}{2},y} - (\hat{E}_{y,h}\Phi^+_{3,h})_{i-\frac{1}{2},y}]dy \\
&- \int_{K_{i,j}} E_{y,h}\partial_x\Phi_{3,h} - \int_{I_i} [(\hat{E}_{x,h}\Phi^-_{3,h})_{x,j+\frac{1}{2}} - (\hat{E}_{x,h}\Phi^+_{3,h})_{x,j-\frac{1}{2}}]dx \\
&+ \int_{K_{i,j}} E_{x,h}\partial_y\Phi_{3,h} = 0,
\end{aligned} \tag{4.26}$$

$$\int_{K_{i,j}} \partial_t D_{x,h}\Phi_{1,h} = \varepsilon_0(1+\chi^{(1)})\int_{K_{i,j}} \partial_t E_{x,h}\Phi_{1,h} + \varepsilon_0\chi^{(3)}\int_{K_{i,j}} \Big[\big(E_{x,h}^2 + E_{y,h}^2\big)\partial_t E_{x,h}\Phi_{1,h} + 2\big(E_{x,h}^2\partial_t E_{x,h}\Phi_{1,h} + E_{x,h}E_{y,h}\partial_t E_{y,h}\Phi_{1,h}\big)\Big], \tag{4.27}$$

$$\int_{K_{i,j}} \partial_t D_{y,h}\Phi_{2,h} = \varepsilon_0(1+\chi^{(1)})\int_{K_{i,j}} \partial_t E_{y,h}\Phi_{2,h} + \varepsilon_0\chi^{(3)}\int_{K_{i,j}} \Big[\big(E_{x,h}^2 + E_{y,h}^2\big)\partial_t E_{y,h}\Phi_{2,h} + 2\big(E_{y,h}^2\partial_t E_{y,h}\Phi_{2,h} + E_{x,h}E_{y,h}\partial_t E_{x,h}\Phi_{2,h}\big)\Big]. \tag{4.28}$$

The initial conditions are defined as

$$E_{x,h}(x,0) = E_{x,h}^0 \quad \text{and} \quad E_{y,h}(y,0) = E_{y,h}^0 \quad \text{and} \quad H_{z,h}(z,0) = E_{z,h}^0,$$

and the concrete choice of the discrete initial data $(E_{x,h}^0, E_{y,h}^0, H_{z,h}^0) \in U_h^k$ will be given later.

4.1.1 The Nonlinear Electromagnetic Energy of Discontinuous Galerkin Method at the Semi-Discrete Level

The nonlinear electromagnetic energy of the discontinuous Galerkin method at the semi-discrete level of the system (4.24)–(4.28) at time t is defined by

$$\begin{aligned}\mathcal{E}_h(t) := &\Big[\|E_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})}^2 + \|E_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}^2\Big] + \|H_{z,h}\|_{\mu_0}^2 \\ &+ \frac{3}{2}\Big\|E_{x,h}^2 + E_{y,h}^2\Big\|_{\varepsilon_0\chi^{(3)}}^2 + 2c_0\int_0^t \Big[\int_r^s (E_{x,h}^+)_{x,\frac{1}{2}}^2 dx + \int_p^q (E_{y,h}^+)_{\frac{1}{2},y}^2 dy\Big].\end{aligned}$$

In the next, we will show that the nonlinear electromagnetic energy at the semi-discrete level of the system (4.24)–(4.28) at time t is conserved and bounded.

Theorem 4.3 *Given* $(J_{x,h}, J_{y,h}) \in C(0,T,U_h^k)$, *let* $(E_{x,h}, E_{y,h}, H_{z,h}) \in C^1(0,T,U_h^k)$ *be the corresponding finite element solution at the semi-discrete level of the system (4.24)–(4.28). Then the nonlinear electromagnetic energy of the system (4.24)–(4.28) for a vanishing current density at any time* t

satisfies

$$
\begin{aligned}
&\Big[\|E_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big] + \|H_{z,h}\|^2_{\mu_0} \\
&+ \frac{3}{2}\Big\|E^2_{x,h} + E^2_{y,h}\Big\|^2_{\varepsilon_0\chi^{(3)}} + 2c_0\int_0^t \Big[\int_r^s (E^+_{x,h})^2_{x,\frac{1}{2}}dx + \int_p^q (E^+_{y,h})^2_{\frac{1}{2},y}dy\Big] \\
&= \Big[\|E^0_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E^0_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big] + \mu_0\|H^0_{z,h}\|^2 \\
&+ \varepsilon_0\chi^{(3)}\frac{3}{2}\Big\|(E^0_{x,h})^2 + (E^0_{y,h})^2\Big\|^2, \qquad (4.29)
\end{aligned}
$$

and for non-zero current density

$$
\begin{aligned}
&\Big[\|E_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\Big] + \|H_{z,h}\|_{\mu_0} \\
&+ \frac{3}{2}\Big\|E^2_{x,h} + E^2_{y,h}\Big\|_{\varepsilon_0\chi^{(3)}} + 2c_0\int_0^t \Big[\int_r^s (E^+_{x,h})_{x,\frac{1}{2}}dx + \int_p^q (E^+_{y,h})^2_{\frac{1}{2},y}dy\Big] \\
&\leq C\Big[\Big[\|E^0_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})} + \|E^0_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\Big] + \mu_0\|H^0_{z,h}\| \\
&+ \varepsilon_0\chi^{(3)}\frac{3}{2}\Big\|(E^0_{x,h})^2 + (E^0_{y,h})^2\Big\| + 2\int_0^t \|J_{x,h} + J_{y,h}\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}ds\Big]. \qquad (4.30)
\end{aligned}
$$

where $C > 0$ is a constant independent of t and h.

Proof: Taking $\Phi_{1,h} = E_{x,h}$ in the equations (4.24) and (4.27), and substituting the equation (4.27) into the equation (4.24), we have

$$
\begin{aligned}
&\varepsilon_0(1+\chi^{(1)})\int_{K_{i,j}} \partial_t E_{x,h}E_{x,h} + \varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[\big(E^2_{x,h} + E^2_{y,h}\big)\partial_t E_{x,h}E_{x,h} \\
&+ 2\big(E^2_{x,h}\partial_t E_{x,h}E_{x,h} + E_{x,h}E_{y,h}\partial_t E_{y,h}E_{x,h}\big)\Big] - \int_{I_i}[(\hat{H}_{z,h}E^-_{x,h})_{x,j+\frac{1}{2}} \\
&- (\hat{H}_{z,h}E^+_{x,h})_{x,j-\frac{1}{2}}]dx + \int_{K_{i,j}} H_{z,h}\partial_y E_{x,h} - \int_{K_{i,j}} J_{x,h}E_{x,h} = 0. \qquad (4.31)
\end{aligned}
$$

Taking $\Phi_{2,h} = E_{y,h}$ in the equations (4.25) and (4.28), and substituting the equation (4.28) into the equation (4.25), we have

$$
\begin{aligned}
&\varepsilon_0(1+\chi^{(1)})\int_{K_{i,j}} \partial_t E_{y,h}E_{y,h} + \varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[\big(E^2_{x,h} + E^2_{y,h}\big)\partial_t E_{y,h}E_{y,h} \\
&+ 2\big(E^2_{y,h}\partial_t E_{y,h}E_{y,h} + E_{x,h}E_{y,h}\partial_t E_{x,h}E_{y,h}\big)\Big] + \int_{J_j}[(\hat{H}_{z,h}E^-_{y,h})_{i+\frac{1}{2},y} \\
&- (\hat{H}_{z,h}E^+_{y,h})_{i-\frac{1}{2},y}]dy - \int_{K_{i,j}} H_{z,h}\partial_x E_{y,h} - \int_{K_{i,j}} J_{y,h}E_{y,h} = 0. \qquad (4.32)
\end{aligned}
$$

Adding the equations (4.31) and (4.32), we have

$$
\begin{aligned}
&\varepsilon_0(1+\chi^{(1)})\frac{1}{2}\frac{d}{dt}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{x,h}^2\Big]+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{y,h}^2\Big]\frac{1}{2}\frac{d}{dt}\Big[E_{x,h}^2\\
&+E_{x,h}^2\Big]+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}2\Big[E_{x,h}^2\frac{1}{2}\frac{d}{dt}E_{x,h}^2+E_{y,h}^2\frac{1}{2}\frac{d}{dt}E_{y,h}^2\Big]\\
&+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}2\Big[E_{x,h}E_{y,h}[\frac{d}{dt}E_{y,h}E_{x,h}+\frac{d}{dt}E_{x,h}E_{y,h}]\Big]\\
&-\int_{I_i}[(\hat{H}_{z,h}E_{x,h}^-)_{x,j+\frac{1}{2}}-(\hat{H}_{z,h}E_{x,h}^+)_{x,j-\frac{1}{2}}]dx+\int_{J_j}[(\hat{H}_{z,h}E_{y,h}^-)_{i+\frac{1}{2},y}\\
&-(\hat{H}_{z,h}E_{y,h}^+)_{i-\frac{1}{2},y}]dy+\int_{K_{i,j}}H_{z,h}\partial_y E_{x,h}-\int_{K_{i,j}}H_{z,h}\partial_x E_{y,h}-\int_{K_{i,j}}J_{x,h}E_{x,h}\\
&-\int_{K_{i,j}}J_{y,h}E_{y,h}=0.
\end{aligned}
\tag{4.33}
$$

Taking $\Phi_{3,h}=H_{z,h}$ in the equations (4.26), we have

$$
\begin{aligned}
&\mu_0\int_{K_{i,j}}\partial_t H_{z,h}H_{z,h}+\int_{J_j}[(\hat{E}_{y,h}H_{z,h}^-)_{i+\frac{1}{2},y}-(\hat{E}_{y,h}H_{z,h}^+)_{i-\frac{1}{2},y}]dy\\
&-\int_{K_{i,j}}E_{y,h}\partial_x H_{z,h}-\int_{I_i}[(\hat{E}_{x,h}H_{z,h}^-)_{x,j+\frac{1}{2}}-(\hat{E}_{x,h}H_{z,h}^+)_{x,j-\frac{1}{2}}]dx\\
&+\int_{K_{i,j}}E_{x,h}\partial_y H_{z,h}=0.
\end{aligned}
\tag{4.34}
$$

Adding the equations (4.33) and (4.34), we have

$$
\begin{aligned}
&\varepsilon_0(1+\chi^{(1)})\frac{1}{2}\frac{d}{dt}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{x,h}^2\Big]+\mu_0\frac{1}{2}\frac{d}{dt}\int_{K_{i,j}}H_{z,h}^2\\
&+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{y,h}^2\Big]\frac{1}{2}\frac{d}{dt}\Big[E_{x,h}^2+E_{x,h}^2\Big]\\
&+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}2\Big[E_{x,h}^2\frac{1}{2}\frac{d}{dt}E_{x,h}^2+E_{y,h}^2\frac{1}{2}\frac{d}{dt}E_{y,h}^2\Big]\\
&+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}2\Big[E_{x,h}E_{y,h}[\frac{d}{dt}E_{y,h}E_{x,h}+\frac{d}{dt}E_{x,h}E_{y,h}]\Big]-\int_{I_i}[(\hat{H}_{z,h}E_{x,h}^-)_{x,j+\frac{1}{2}}\\
&-(\hat{H}_{z,h}E_{x,h}^+)_{x,j-\frac{1}{2}}]dx+\int_{J_j}[(\hat{H}_{z,h}E_{y,h}^-)_{i+\frac{1}{2},y}-(\hat{H}_{z,h}E_{y,h}^+)_{i-\frac{1}{2},y}]dy\\
&+\int_{J_j}[(\hat{E}_{y,h}H_{z,h}^-)_{i+\frac{1}{2},y}-(\hat{E}_{y,h}H_{z,h}^+)_{i-\frac{1}{2},y}]dy-\int_{I_i}[(\hat{E}_{x,h}H_{z,h}^-)_{x,j+\frac{1}{2}}\\
&-(\hat{E}_{x,h}H_{z,h}^+)_{x,j-\frac{1}{2}}]dx+\int_{K_{i,j}}H_{z,h}\partial_y E_{x,h}-\int_{K_{i,j}}H_{z,h}\partial_x E_{y,h}-\int_{K_{i,j}}E_{y,h}\partial_x H_{z,h}\\
&+\int_{K_{i,j}}E_{x,h}\partial_y H_{z,h}-\int_{K_{i,j}}J_{x,h}E_{x,h}-\int_{K_{i,j}}J_{y,h}E_{y,h}=0. \qquad (4.35)
\end{aligned}
$$

Now we rewrite the nonlinear terms from the left hand side of equation (4.35):

$$
\begin{aligned}
&\varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{y,h}^2\Big]\frac{1}{2}\frac{d}{dt}\Big[E_{x,h}^2+E_{x,h}^2\Big]\\
&\quad+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}2\Big[E_{x,h}^2\frac{1}{2}\frac{d}{dt}E_{x,h}^2+E_{y,h}^2\frac{1}{2}\frac{d}{dt}E_{y,h}^2\Big]\\
&\quad+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}2\Big[E_{x,h}E_{y,h}\frac{d}{dt}E_{y,h}E_{x,h}+E_{x,h}E_{y,h}\frac{d}{dt}E_{x,h}E_{y,h}\Big]\\
=&\varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{y,h}^2\Big]\frac{1}{2}\frac{d}{dt}\Big[E_{x,h}^2+E_{x,h}^2\Big]\\
&\quad+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}2\Big[E_{x,h}^2\frac{1}{2}\frac{d}{dt}E_{x,h}^2+E_{y,h}^2\frac{1}{2}\frac{d}{dt}E_{y,h}^2\Big]\\
&\quad+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}2\Big[E_{x,h}^2\frac{1}{2}\frac{d}{dt}E_{y,h}^2+E_{y,h}^2\frac{1}{2}\frac{d}{dt}E_{x,h}^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
=&\varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{y,h}^2\Big]\frac{1}{2}\frac{d}{dt}\Big[E_{x,h}^2+E_{x,h}^2\Big]\\
&+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[\Big[E_{x,h}^2+E_{y,h}^2\Big]\frac{d}{dt}E_{x,h}^2+\Big[E_{x,h}^2+E_{y,h}^2\Big]\frac{d}{dt}E_{y,h}^2\Big]\\
=&\varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{y,h}^2\Big]\frac{1}{2}\frac{d}{dt}\Big[E_{x,h}^2+E_{x,h}^2\Big]\\
&+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{y,h}^2\Big]\frac{d}{dt}\Big[E_{x,h}^2+E_{x,h}^2\Big]\\
=&\varepsilon_0\chi^{(3)}\frac{3}{2}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{y,h}^2\Big]\frac{d}{dt}\Big[E_{x,h}^2+E_{x,h}^2\Big]\\
=&\varepsilon_0\chi^{(3)}\frac{3}{4}\int_{K_{i,j}}\frac{d}{dt}\Big[E_{x,h}^2+E_{x,h}^2\Big]^2.
\end{aligned}
\tag{4.36}
$$

This relation $\Big[E_{x,h}^2+E_{y,h}^2\Big]\frac{d}{dt}\Big[E_{x,h}^2+E_{y,h}^2\Big]=\frac{1}{2}\frac{d}{dt}\Big[E_{x,h}^2+E_{y,h}^2\Big]^2$ is used to simplify the term in equation (4.36). We substitute the equation (4.36) into equation (4.35):

$$
\begin{aligned}
&\varepsilon_0(1+\chi^{(1)})\frac{1}{2}\frac{d}{dt}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{x,h}^2\Big]+\mu_0\frac{1}{2}\frac{d}{dt}\int_{K_{i,j}}H_{z,h}^2\\
&+\varepsilon_0\chi^{(3)}\frac{3}{4}\frac{d}{dt}\int_{K_{i,j}}\Big[E_{x,h}^2+E_{y,h}^2\Big]^2-\int_{I_i}[(\hat{H}_{z,h}E_{x,h}^-)_{x,j+\frac{1}{2}}\\
&-(\hat{H}_{z,h}E_{x,h}^+)_{x,j-\frac{1}{2}}]dx+\int_{J_j}[(\hat{H}_{z,h}E_{y,h}^-)_{i+\frac{1}{2},y}-(\hat{H}_{z,h}E_{y,h}^+)_{i-\frac{1}{2},y}]dy\\
&+\int_{J_j}[(\hat{E}_{y,h}H_{z,h}^-)_{i+\frac{1}{2},y}-(\hat{E}_{y,h}H_{z,h}^+)_{i-\frac{1}{2},y}]dy-\int_{I_i}[(\hat{E}_{x,h}H_{z,h}^-)_{x,j+\frac{1}{2}}\\
&-(\hat{E}_{x,h}H_{z,h}^+)_{x,j-\frac{1}{2}}]dx+\int_{K_{i,j}}H_{z,h}\partial_yE_{x,h}-\int_{K_{i,j}}H_{z,h}\partial_xE_{y,h}-\int_{K_{i,j}}E_{y,h}\partial_xH_{z,h}\\
&+\int_{K_{i,j}}E_{x,h}\partial_yH_{z,h}-\int_{K_{i,j}}J_{x,h}E_{x,h}-\int_{K_{i,j}}J_{y,h}E_{y,h}=0.
\end{aligned}
\tag{4.37}
$$

The following estimates are well-known, for details see [93, equations (3.18)-(3.19)]:

$$\sum_{1\le j\le N_y}\Big[-\int_{I_i}[(\hat{H}_{z,h}E^-_{x,h})_{x,j+\frac{1}{2}}-(\hat{H}_{z,h}E^+_{x,h})_{x,j-\frac{1}{2}}]dx-\int_{I_i}[(\hat{E}_{x,h}H^-_{z,h})_{x,j+\frac{1}{2}}$$
$$-(\hat{E}_{x,h}H^+_{z,h})_{x,j-\frac{1}{2}}]dx+\int_{K_{i,j}}H_{z,h}\partial_y E_{x,h}+\int_{K_{i,j}}E_{x,h}\partial_y H_{z,h}\Big]$$
$$=c_0\int_{I_i}(E^+_{x,h})^2_{x,\frac{1}{2}}dx. \tag{4.38}$$

and

$$\sum_{1\le i\le N_x}\Big[\int_{J_j}[(\hat{H}_{z,h}E^-_{y,h})_{i+\frac{1}{2},y}-(\hat{H}_{z,h}E^+_{y,h})_{i-\frac{1}{2},y}]dy+\int_{J_j}[(\hat{E}_{y,h}H^-_{z,h})_{i+\frac{1}{2},y}$$
$$-(\hat{E}_{y,h}H^+_{z,h})_{i-\frac{1}{2},y}]dy-\int_{K_{i,j}}H_{z,h}\partial_x E_{y,h}-\int_{K_{i,j}}E_{y,h}\partial_x H_{z,h}\Big]$$
$$=c_0\int_{J_j}(E^+_{y,h})^2_{\frac{1}{2},y}dy. \tag{4.39}$$

Finally summing the equations (4.37) with respect to both indexes $1\le i\le N_x$ and $1\le j\le N_y$ and using the estimates (4.38)–(4.39), we get

$$\frac{1}{2}\frac{d}{dt}\Big[\|E_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|E_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big]+\frac{1}{2}\frac{d}{dt}\|H_{z,h}\|^2_{\mu_0}$$
$$+\frac{3}{4}\frac{d}{dt}\Big\|E^2_{x,h}+E^2_{y,h}\Big\|^2_{\varepsilon_0\chi^{(3)}}+c_0\int_r^s(E^+_{x,h})^2_{x,\frac{1}{2}}dx+c_0\int_p^q(E^+_{y,h})^2_{\frac{1}{2},y}dy$$
$$=\int_\Omega[J_{x,h}E_{x,h}+J_{y,h}E_{y,h}]. \tag{4.40}$$

The right-hand side of the equation (4.40) is figured out by employing the inequality (3) from Lemma 5.1. This gives

$$\int_\Omega[J_{x,h}E_{x,h}+J_{y,h}E_{y,h}]$$
$$=\int_\Omega\varepsilon_0(1+\chi^{(1)})[E_{x,h}+E_{y,h}]\cdot\Big(\varepsilon_0(1+\chi^{(1)})\Big)^{-1}[J_{x,h}+J_{y,h}]$$
$$\le\|[E_{x,h}+E_{y,h}]\|_{\varepsilon_0(1+\chi^{(1)})}\,\|[J_{x,h}+J_{y,h}]\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}. \tag{4.41}$$

Then we get from equation (4.40)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\|E_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|E_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big]+\frac{1}{2}\frac{d}{dt}\|H_{z,h}\|^2_{\mu_0}\\
&+\frac{3}{4}\frac{d}{dt}\Big\|E^2_{x,h}+E^2_{y,h}\Big\|^2_{\varepsilon_0\chi^{(3)}}+c_0\int_r^s(E^+_{x,h})^2_{x,\frac{1}{2}}dx+c_0\int_p^q(E^+_{y,h})^2_{\frac{1}{2},y}dy\\
&\le\|[E_{x,h}+E_{y,h}]\|_{\varepsilon_0(1+\chi^{(1)})}\,\|[J_{x,h}+J_{y,h}]\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}.
\end{aligned}
$$

Integrating both sides from 0 to t

$$
\begin{aligned}
&\frac{1}{2}\Big[\|E_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|E_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big]+\frac{1}{2}\|H_{z,h}\|^2_{\mu_0}\\
&+\frac{3}{4}\Big\|E^2_{x,h}+E^2_{y,h}\Big\|^2_{\varepsilon_0\chi^{(3)}}+c_0\int_0^t\Big[\int_r^s(E^+_{x,h})^2_{x,\frac{1}{2}}dx+\int_p^q(E^+_{y,h})^2_{\frac{1}{2},y}dy\Big]\\
&-\frac{1}{2}\Big[\|E^0_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|E^0_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big]-\mu_0\frac{1}{2}\|H^0_{z,h}\|^2\\
&-\varepsilon_0\chi^{(3)}\frac{3}{4}\Big\|(E^0_{x,h})^2+(E^0_{y,h})^2\Big\|^2\\
&\le\int_0^t\|E_{x,h}+E_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\,\|J_{x,h}+J_{y,h}\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}ds\\
&\le\sqrt{2}\int_0^t\sqrt{\|E_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|E_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}}\,\|J_{x,h}+J_{y,h}\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}ds.
\end{aligned}
$$

Then it follows from the Gronwall–Ou-Iang's inequality (see, e.g., [114])

$$
\begin{aligned}
&\frac{1}{2}\Big[\|E_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})}+\|E_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\Big]+\frac{1}{2}\|H_{z,h}\|_{\mu_0}\\
&+\frac{3}{4}\Big\|E^2_{x,h}+E^2_{y,h}\Big\|_{\varepsilon_0\chi^{(3)}}+c_0\int_0^t\Big[\int_r^s(E^+_{x,h})_{x,\frac{1}{2}}dx+\int_p^q(E^+_{y,h})^2_{\frac{1}{2},y}dy\Big]\\
&\le C\Big[\frac{1}{2}\Big[\|E^0_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})}+\|E^0_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\Big]+\mu_0\frac{1}{2}\|H^0_{z,h}\|+\varepsilon_0\chi^{(3)}\frac{3}{4}\Big\|(E^0_{x,h})^2\\
&+(E^0_{y,h})^2\Big\|+\int_0^t\|J_{x,h}+J_{y,h}\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}ds\Big],
\end{aligned}
$$

which completes the proof of (4.30). Similarly the result (4.29) can be obtained for the case $(J_{x,h},J_{y,h})=(0,0)$. $\square$

4.2 Error Estimates for the Semi-Discrete Problem Using the Discontinuous Galerkin Method

Let $P_k(I_i)$ be the kth degree polynomial space over the interval I_i. 1D projection operators $P_x^{\pm}$ are employed that are frequently used in DG (discontinuous Galerkin) and LDG (local discontinuous Galerkin) methods [34, 46]. For any function $u \in H(I_i) := H^1(I_i)$, we define

$$P_x^{\pm} : H(I_i) \to P_k(I_i),$$

and the operators satisfy

$$\begin{aligned}
&\int_{I_i} (P_x^{+} u) w = \int_{I_i} u\, w \quad \forall w \in P_{k-1}(I_i) \quad \text{and} \quad P_x^{+} u\Big(x^{+}_{i-\frac{1}{2}}\Big) = u\Big(x^{+}_{i-\frac{1}{2}}\Big),\\
&\int_{I_i} (P_x^{-} u) w = \int_{I_i} u\, w \quad \forall w \in P_{k-1}(I_i) \quad \text{and} \quad P_x^{-} u\Big(x^{-}_{i+\frac{1}{2}}\Big) = u\Big(x^{-}_{i+\frac{1}{2}}\Big).
\end{aligned}$$

Moreover, for any function $u \in H(J_j)$, the $P_y^{\pm}$ projection operators in y-direction are defined as

$$P_y^{\pm} : H(J_j) \to P_k(J_j),$$

and satisfy

$$\begin{aligned}
&\int_{J_j} (P_y^{+} u) w = \int_{J_j} u\, w \quad \forall w \in P_{k-1}(J_j) \quad \text{and} \quad P_y^{+} u\Big(y^{+}_{j-\frac{1}{2}}\Big) = u\Big(y^{+}_{j-\frac{1}{2}}\Big),\\
&\int_{J_j} (P_y^{-} u) w = \int_{J_j} u\, w \quad \forall w \in P_{k-1}(J_j) \quad \text{and} \quad P_y^{-} u\Big(y^{-}_{j+\frac{1}{2}}\Big) = u\Big(y^{-}_{j+\frac{1}{2}}\Big).
\end{aligned}$$

The standard local L^2 projection operators in 1D are denoted by

$$P_x : H(I_i) \to P_k(I_i), \quad \text{and} \quad P_y : H(J_j) \to P_k(J_j).$$

The projection operators for rectangular elements $K_{i,j} = I_i \times J_j$ in 2D are defined as tensor products of the 1D projections. We define

$$\Pi_1 := P_x \otimes P_y^{+} : H^2(K_{i,j}) \to Q_k(K_{i,j}), \tag{4.42}$$

which satisfies

$$\int_{K_{i,j}} [\Pi_1 w(x,y) \partial_y u_h(x,y)] dxdy = \int_{K_{i,j}} [w(x,y) \partial_y u_h(x,y)] dxdy, \tag{4.43}$$

$$\int_{I_i} \Pi_1 w\Big(x, y^{+}_{j-\frac{1}{2}}\Big) u_h\Big(x, y^{+}_{j-\frac{1}{2}}\Big) dx = \int_{I_i} w\Big(x, y^{+}_{j-\frac{1}{2}}\Big) u_h\Big(x, y^{+}_{j-\frac{1}{2}}\Big) dx \tag{4.44}$$

for all $w \in H^2(K_{i,j})$ and $u_h \in Q_k(K_{i,j})$ [101, 93]. The projection Π_2 is defined as

$$\Pi_2 := P_x^+ \otimes P_y : H^2(K_{i,j}) \to Q_k(K_{i,j}), \tag{4.45}$$

which satisfies

$$\int_{K_{i,j}} [\Pi_2 w(x,y)\partial_x u_h(x,y)]dxdy = \int_{K_{i,j}} [w(x,y)\partial_x u_h(x,y)]dxdy, \tag{4.46}$$

$$\int_{J_j} \Pi_2 w\Big(x^+_{i-\frac{1}{2}}, y\Big) u_h\Big(x^+_{i-\frac{1}{2}}, y\Big) dy = \int_{J_j} w\Big(x^+_{i-\frac{1}{2}}, y\Big) u_h\Big(x^+_{i-\frac{1}{2}}, y\Big) dy \tag{4.47}$$

for all $w \in H^2(K_{i,j})$ and $u_h \in Q_k(K_{i,j})$ [101, 93]. The projection Π_3 is defined as

$$\Pi_3 := P_x^- \otimes P_y^- : H^2(K_{i,j}) \to Q_k(K_{i,j}), \tag{4.48}$$

which satisfies

$$\int_{K_{i,j}} [\Pi_3 w(x,y) u_h(x,y)]dxdy = \int_{K_{i,j}} [w(x,y) u_h(x,y)]dxdy, \tag{4.49}$$

$$\int_{I_i} \Pi_3 w\Big(x, y^-_{j+\frac{1}{2}}\Big) u_h\Big(x, y^-_{j+\frac{1}{2}}\Big) dx = \int_{I_i} w\Big(x, y^-_{j+\frac{1}{2}}\Big) u_h\Big(x, y^-_{j+\frac{1}{2}}\Big) dx, \tag{4.50}$$

$$\int_{J_j} \Pi_3 w\Big(x^-_{i+\frac{1}{2}}, y\Big) u_h\Big(x^-_{i+\frac{1}{2}}, y\Big) dy = \int_{J_j} w\Big(x^-_{i+\frac{1}{2}}, y\Big) u_h\Big(x^-_{i+\frac{1}{2}}, y\Big) dy, \tag{4.51}$$

$$\Pi_3 w\Big(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}\Big) = w\Big(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}\Big). \tag{4.52}$$

for all $w \in H^2(K_{i,j})$ and $u_h \in Q_{k-1}(K_{i,j})$ [101, 93]. The use of the H^2 spaces for the point values makes sense by the Sobolev embedding $H^2 \subset C^0$ in 2D. The L^2 projection operator is usually defined by

$$\Pi_4 := P_x \otimes P_y : H^2(K_{i,j}) \to Q_k(K_{i,j}), \tag{4.53}$$

for the properties see [34, 46] [93, equations (3.33)–(3.42)].

Lemma 4.4 *If w is a product of 1D functions, i.e $w(x,y) = f(x)g(y)$, where $f \in H(I_i)$ and $g \in H(J_j)$, we have*

$$\Pi_1 w(x,y) = P_x f(x) P_y^+ g(y) \quad , \qquad \Pi_2 w(x,y) = P_x^+ f(x) P_y g(y),$$
$$\Pi_3 w(x,y) = P_x^- f(x) P_y^- g(y) \quad , \qquad \Pi_4 w(x,y) = P_x f(x) P_y g(y).$$

These results demonstrate that the 2D projection is a tensor product of 1D projections, for details see [34, 46].

Lemma 4.5 *For the projection operators $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ defined in (4.42), (4.45), (4.48) and (4.53), respectively, there exists a constant $C > 0$ independent of h such that for all $u \in H^{k+1}(\Omega)$ and $k \geq 1$*

$$\|\Pi_i u - u\| \leq C h^{k+1} \|u\|_{H^{k+1}(\Omega)}, \quad \forall \quad i = 1 \ldots 4. \tag{4.54}$$

Remark 4.6 These projection errors can be estimated by means of (4.54), that is, for $E_x, \partial_t E_x, E_y, \partial_t E_y$ and $H_z, \partial_t H_z$, we have that

$$\begin{aligned}\|E_x - \Pi_1 E_x\|_{\varepsilon_0} &\leq C\sqrt{\varepsilon_0}\, h^k \|E_x\|_{H^{k+1}(\Omega)} \leq C\sqrt{\varepsilon_0}\, h^k \|E_x\|_{\mathbf{C}(0,T,H^{k+1}(\Omega))}, \\ \|\partial_t E_x - \Pi_1 \partial_t E_x\|_{\varepsilon_0} &\leq C\sqrt{\varepsilon_0}\, h^k \|\partial_t E_x\|_{H^{k+1}(\Omega)} \leq C\sqrt{\varepsilon_0}\, h^k \|\partial_t E_x\|_{\mathbf{C}(0,T,H^{k+1}(\Omega))},\end{aligned} \tag{4.55}$$

$$\begin{aligned}\|E_y - \Pi_2 E_y\|_{\varepsilon_0} &\leq C\sqrt{\varepsilon_0}\, h^k \|E_y\|_{H^{k+1}(\Omega)} \leq C\sqrt{\varepsilon_0}\, h^k \|E_y\|_{\mathbf{C}(0,T,H^{k+1}(\Omega))}, \\ \|\partial_t E_y - \Pi_2 \partial_t E_y\|_{\varepsilon_0} &\leq C\sqrt{\varepsilon_0}\, h^k \|\partial_t E_y\|_{H^{k+1}(\Omega)} \leq C\sqrt{\varepsilon_0}\, h^k \|\partial_t E_y\|_{\mathbf{C}(0,T,H^{k+1}(\Omega))},\end{aligned} \tag{4.56}$$

$$\|\partial_t H_z - \Pi_3 \partial_t H_z\|_{\mu_0} \leq C\sqrt{\mu_0}\, h^k \|\partial_t H_z\|_{H^{k+1}(\Omega)} \leq C\sqrt{\mu_0}\, h^k \|\partial_t H_z\|_{\mathbf{C}(0,T,H^{k+1}(\Omega))}. \tag{4.57}$$

Let (E_x, E_y, H_z) be the weak solution of (4.6)–(4.10) and $(E_{x,h}, E_{y,h}, H_{z,h})$ be corresponding numerical solution of the semi-discrete scheme (4.24)–(4.28). We denote the error terms for later use by

$$\zeta_x := E_x - E_{x,h} = \eta_x - \eta_{x,h}, \tag{4.58}$$

where

$$\eta_x := E_x - \Pi_1 E_x, \quad \eta_{x,h} := E_{x,h} - \Pi_1 E_x. \tag{4.59}$$

Similarly for the electric field in y-direction we set

$$\zeta_y := E_y - E_{y,h} = \eta_y - \eta_{y,h}, \tag{4.60}$$

where

$$\eta_y := E_y - \Pi_2 E_y, \quad \eta_{y,h} := E_{y,h} - \Pi_2 E_y. \tag{4.61}$$

The error terms for the magnetic field are defined by:

$$\xi_z := H_z - H_{z,h} = \theta_z - \theta_{z,h}, \tag{4.62}$$

where

$$\theta_z := H_z - \Pi_3 H_z, \quad \theta_{z,h} := H_{z,h} - \Pi_3 H_z. \tag{4.63}$$

Lemma 4.7 *There exists a constant $C > 0$ independent of h such that*

$$\sum_{1\le i\le N_x} \Big[\int_{I_i} [-(\hat{\theta}_z\eta^-_{x,h})_{x,j+\frac{1}{2}} - (\hat{\theta}_z\eta^+_{x,h})_{x,j-\frac{1}{2}}] + \int_{K_{i,j}} \theta_z\partial_y\eta_{x,h}\Big]$$
$$\le Ch^{2k+2} + \|\eta_{x,h}\|^2, \tag{4.64}$$
$$\sum_{1\le j\le N_y} \Big[\int_{J_j} [(\hat{\theta}_z\eta^-_{y,h})_{i+\frac{1}{2},y} - (\hat{\theta}_z\eta^+_{y,h})_{j-\frac{1}{2},y}] - \int_{K_{i,j}} \theta_z\partial_x\eta_{y,h}\Big]$$
$$\le Ch^{2k+2} + \|\eta_{y,h}\|^2. \tag{4.65}$$

Proof: The details of the proof can be found in [93, Lemma 3.4]. □

Lemma 4.8 *There exists a constant $C > 0$ independent of h such that*

$$\sum_{1\le i\le N_x} \Big[\int_{I_i} [-(\hat{\theta}_z\eta^-_{x,h})_{x,j+\frac{1}{2}} - (\hat{\theta}_z\eta^+_{x,h})_{x,j-\frac{1}{2}}] + \int_{K_{i,j}} \theta_z\partial_y\eta_{x,h}\Big]$$
$$- \sum_{1\le i\le N_x} c_0 \int_{I_i} [\eta^+_{x,h}(x, y_{\frac{1}{2}})]^2 \le Ch^{2k+2} + \|\eta_{x,h}\|^2, \tag{4.66}$$
$$\sum_{1\le j\le N_y} \Big[\int_{J_j} [(\hat{\theta}_z\eta^-_{y,h})_{i+\frac{1}{2},y} - (\hat{\theta}_z\eta^+_{y,h})_{j-\frac{1}{2},y}] - \int_{K_{i,j}} \theta_z\partial_x\eta_{y,h}\Big]$$
$$- \sum_{1\le j\le N_y} c_0 \int_{J_j} [\eta^+_{y,h}(x_{\frac{1}{2}}, y)]^2 \le Ch^{2k+2} + \|\eta_{y,h}\|^2. \tag{4.67}$$

Proof: The details of the proof can be found in [93, Lemma 3.5]. □

Remark 4.9 When $c_0 = 0$, we obtain PEC boundary condition without the jump terms in (4.17) and (4.19) . In this case, we can only control the term $\sum_{1\le i\le N_x}\int_{I_i}(\theta^+_z, \eta^+_{z,h})(x, c)$ as follows

$$\sum_{1\le i\le N_x} \int_{I_i} (\theta^+_z, \eta^+_{z,h})(x, c) \le h^{-1}\int_r^s (\theta^+_z)^2(x, c) + h\int_r^s (\eta^+_{z,h})^2(x, c)$$
$$\le Ch^{2k+1} + h\|\eta_{z,h}\|^2, \tag{4.68}$$

by an inverse inequality. Therefore we lose half an order.

Theorem 4.10 *Suppose that the weak solution (E_x, E_y, H_z) of the system (4.6)–(4.10), and the finite element solution $(E_{x,h}, E_{y,h}, H_{z,h})$ of the system*

(4.24)–(4.28), respectively, exist. Then the following error estimate holds with a constant $C > 0$ independent of h and t such that

$$\|E_x - E_{x,h}\|_{\varepsilon_0} + \|E_y - E_{y,h}\|_{\varepsilon_0} + \|H_z - H_{z,h}\|_{\mu_0} \leq Ch^k.$$

Proof: Subtracting the equations (4.24)–(4.28) from the equations (4.6)–(4.10), using the error identities (4.58), (4.60), and (4.62), for all test functions Φ_{1h}, Φ_{2h} $\Phi_{3h} \in Q_k(K_{i,j})$, we obtain

$$\begin{aligned}
&\int_{K_{i,j}} \partial_t(D_x - D_{x,h})\Phi_{1,h} - \int_{I_i} [(\hat{\xi}_z \Phi^-_{1,h})_{x,j+\frac{1}{2}} - (\hat{\xi}_z \Phi^+_{1,h})_{x,j-\frac{1}{2}}]dx \\
&+ \int_{K_{i,j}} \xi_z \partial_y \Phi_{1,h} = 0, \qquad (4.69)\\
&\int_{K_{i,j}} \partial_t(D_y - D_{y,h})\Phi_{2,h} + \int_{J_j} [(\hat{\xi}_z \Phi^-_{2,h})_{i+\frac{1}{2},y} - (\hat{\xi}_z \Phi^+_{2,h})_{i-\frac{1}{2},y}]dy \\
&- \int_{K_{i,j}} \xi_z \partial_x \Phi_{2,h} = 0, \qquad (4.70)\\
&\mu_0 \int_{K_{i,j}} \partial_t \xi_z \Phi_{3,h} + \int_{J_j} [(\hat{\zeta}_y \Phi^-_{3,h})_{i+\frac{1}{2},y} - (\hat{\zeta}_y \Phi^+_{3,h})_{i-\frac{1}{2},y}]dy \\
&- \int_{K_{i,j}} \zeta_y \partial_x \Phi_{3,h} - \int_{I_i} [(\hat{\zeta}_x \Phi^-_{3,h})_{x,j+\frac{1}{2}} - (\hat{\zeta}_x \Phi^+_{3,h})_{x,j-\frac{1}{2}}]dx \\
&+ \int_{K_{i,j}} \zeta_x \partial_y \Phi_{3,h} = 0, \qquad (4.71)\\
&\int_{K_{i,j}} \partial_t(D_x - D_{x,h})\Phi_{1,h} = \varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} \partial_t \zeta_x \Phi_{1,h} \\
&+ \varepsilon_0 \chi^{(3)} \int_{K_{i,j}} \Big[[\big(E_x^2 + E_y^2\big) - \big(E_{x,h}^2 + E_{y,h}^2\big)]\partial_t E_x \Phi_{1,h} \\
&+ \big(E_{x,h}^2 + E_{y,h}^2\big)\partial_t[E_x - E_{x,h}]\Phi_{1,h} + 2\big([E_x^2 - E_{x,h}^2]\partial_t E_x \Phi_{1,h} \\
&+ [E_x E_y - E_{x,h}E_{y,h}]\partial_t E_y \Phi_{1,h}\big) + 2\big(E_{x,h}^2 \partial_t[E_x - E_{x,h}]\Phi_{1,h} \\
&+ E_{x,h}E_{y,h}\partial_t[E_y - E_{y,h}]\Phi_{1,h}\big)\Big] = 0, \qquad (4.72)
\end{aligned}$$

$$
\begin{aligned}
&\int_{K_{i,j}} \partial_t(D_y - D_{y,h})\Phi_{2,h} = \varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} \partial_t \zeta_y \Phi_{2,h} \\
&+ \varepsilon_0 \chi^{(3)} \int_{K_{i,j}} \Big[[\big(E_x^2 + E_y^2\big) - \big(E_{x,h}^2 + E_{y,h}^2] \partial_t E_y \Phi_{2,h} \\
&+ \big(E_{x,h}^2 + E_{y,h}^2\big) \partial_t [E_y - E_{y,h}] \Phi_{2,h} + 2\big([E_y^2 - E_{y,h}^2] \partial_t E_y \Phi_{2,h} \\
&+ [E_x E_y - E_{x,h} E_{y,h}] \partial_t E_x \Phi_{2,h}\big) + 2\big(E_{y,h}^2 \partial_t [E_y - E_{y,h}] \Phi_{2,h} \\
&+ E_{x,h} E_{y,h} \partial_t [E_x - E_{x,h}] \Phi_{2,h}\big)\Big] = 0. \qquad (4.73)
\end{aligned}
$$

First we substitute the equations (4.72)–(4.73) into the equations (4.69)–(4.70), respectively. Further decomposing the terms in the resulting equations using (4.59), (4.61) and (4.63) and taking $\Phi_{1,h} = \eta_{x,h}, \Phi_{2,h} = \eta_{y,h}$ and $\Phi_{3,h} = \theta_{z,h}$, we obtain

$$
\begin{aligned}
&\varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} [\partial_t \eta_{x,h}] \eta_{x,h} + \varepsilon_0 \chi^{(3)} \int_{K_{i,j}} \Big[\big(E_{x,h}^2 + E_{y,h}^2\big) [\partial_t \eta_{x,h}] \eta_{x,h} \\
&+ 2E_{x,h}^2 [\partial_t \eta_{x,h}] \eta_{x,h} + 2E_{x,h} E_{y,h} [\partial_t \eta_{y,h}] \eta_{x,h} \Big] \\
&- \int_{I_i} [(\widehat{\theta_{z,h}} \eta_{x,h}^-)_{x,j+\frac{1}{2}} - (\widehat{\theta_{z,h}} \eta_{x,h}^+)_{x,j-\frac{1}{2}}] dx \\
&= \varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} [\partial_t \eta_x] \eta_{x,h} + \varepsilon_0 \chi^{(3)} \int_{K_{i,j}} \big(E_{x,h}^2 + E_{y,h}^2\big) [\partial_t \eta_x] \eta_{x,h} \\
&- \int_{I_i} [(\hat{\theta}_z \eta_{x,h}^-)_{x,j+\frac{1}{2}} - (\hat{\theta}_z \eta_{x,h}^+)_{x,j-\frac{1}{2}}] dx + \int_{K_{i,j}} \theta_z \partial_y \eta_{x,h} \\
&- \int_{K_{i,j}} \theta_{z,h} \partial_y \eta_{x,h} + \varepsilon_0 \chi^{(3)} \int_{K_{i,j}} \Big[[\big(E_x^2 + E_y^2\big) - \big(E_{x,h}^2 + E_{y,h}^2\big)] [\partial_t E_x] \eta_{x,h} \\
&+ 2[E_x^2 - E_{x,h}^2] [\partial_t E_x] \eta_{x,h} + 2[E_x E_y - E_{x,h} E_{y,h}] [\partial_t E_y] \eta_{x,h} \\
&+ 2E_{x,h}^2 [\partial_t \eta_x] \eta_{x,h} + 2E_{x,h} E_{y,h} [\partial_t \eta_y] \eta_{x,h}\big)\Big], \qquad (4.74)
\end{aligned}
$$

and

$$
\begin{aligned}
&\varepsilon_0(1+\chi^{(1)})\int_{K_{i,j}}[\partial_t\eta_{y,h}]\eta_{y,h}+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[\big(E_{x,h}^2+E_{y,h}^2\big)[\partial_t\eta_{y,h}]\eta_{y,h}\\
&+2E_{y,h}^2[\partial_t\eta_{y,h}]\eta_{y,h}+2E_{x,h}E_{y,h}[\partial_t\eta_{x,h}]\eta_{y,h}\Big]\\
&+\int_{J_j}[(\hat{\theta_{z,h}}\eta_{y,h}^-)_{i+\frac{1}{2},y}-(\hat{\theta_{z,h}}\eta_{y,h}^+)_{i-\frac{1}{2},y}]dy\\
&=\varepsilon_0(1+\chi^{(1)})\int_{K_{i,j}}[\partial_t\eta_y\eta_{y,h}+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}\big(E_{x,h}^2+E_{y,h}^2\big)[\partial_t\eta_y]\eta_{y,h}\\
&+\int_{J_j}[(\hat{\theta}_z\eta_{y,h}^-)_{i+\frac{1}{2},y}-(\hat{\theta}_z\eta_{y,h}^+)_{i-\frac{1}{2},y}]dy-\int_{K_{i,j}}\theta_z\partial_x\eta_{y,h}\\
&+\int_{K_{i,j}}\theta_{z,h}\partial_x\eta_{y,h}+\varepsilon_0\chi^{(3)}\int_{K_{i,j}}\Big[[\big(E_x^2+E_y^2\big)-\big(E_{x,h}^2+E_{y,h}^2\big)][\partial_tE_y]\eta_{y,h}\\
&+2[E_y^2-E_{y,h}^2][\partial_tE_y]\eta_{y,h}+2[E_xE_y-E_{x,h}E_{y,h}][\partial_tE_x]\eta_{y,h}\\
&+2E_{y,h}^2[\partial_t\eta_y]\eta_{y,h}+2E_{x,h}E_{y,h}[\partial_t\eta_x]\eta_{y,h}\Big].
\end{aligned}\tag{4.75}
$$

For the magnetic field we have that

$$
\begin{aligned}
&\mu_0\int_{K_{i,j}}[\partial_t\theta_{z,h}]\theta_{z,h}+\int_{J_j}[(\hat{\eta_{y,h}}\theta_{z,h}^-)_{i+\frac{1}{2},y}-(\hat{\eta_{y,h}}\theta_{z,h}^+)_{i-\frac{1}{2},y}]dy\\
&-\int_{I_i}[(\hat{\eta_{x,h}}\theta_{z,h}^-)_{x,j+\frac{1}{2}}-(\hat{\eta_{x,h}}\theta_{z,h}^+)_{x,j-\frac{1}{2}}]dx\\
&=\mu_0\int_{K_{i,j}}[\partial_t\theta_z]\theta_{z,h}+\int_{J_j}[(\hat{\eta_y}\theta_{z,h}^-)_{i+\frac{1}{2},y}-(\hat{\eta_y}\theta_{z,h}^+)_{i-\frac{1}{2},y}]dy\\
&-\int_{K_{i,j}}\eta_y\partial_x\theta_{z,h}+\int_{K_{i,j}}\eta_{y,h}\partial_x\theta_{z,h}-\int_{I_i}[(\hat{\eta_x}\theta_{z,h}^-)_{x,j+\frac{1}{2}}-(\hat{\eta_x}\theta_{z,h}^+)_{x,j-\frac{1}{2}}]dx\\
&+\int_{K_{i,j}}\eta_x\partial_y\theta_{z,h}-\int_{K_{i,j}}\eta_{x,h}\partial_y\theta_{z,h}.
\end{aligned}\tag{4.76}
$$

Adding the equations (4.74)–(4.76) and summing over the indices $1 \le i \le N_x$ and $1 \le j \le N_y$, we obtain the left hand side as

$$\begin{aligned}
LHS =& \frac{1}{2}\frac{d}{dt}[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_{z,h}\|^2_{\mu_0}] \\
&+ \left(E^2_{x,h} + E^2_{y,h}\right)\frac{1}{2}\frac{d}{dt}\Big[\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}} + \|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\Big] \\
&+ 2\Big(E^2_{x,h}\frac{1}{2}\frac{d}{dt}\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}} + E^2_{y,h}\frac{1}{2}\frac{d}{dt}\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\Big) \\
&+ 2\varepsilon_0\chi^{(3)}E_{x,h}E_{y,h}[\partial_t\eta_{y,h}]\eta_{x,h} + 2E_{x,h}E_{y,h}[\partial_t\eta_{x,h}]\eta_{y,h} \\
&+ \sum_{1\le i\le N_x}\int_{I_i}((\hat{\theta_{z,h}} - \theta^+_{z,h})\eta^+_{x,h})(x, y_{\frac{1}{2}}) \\
&+ \sum_{1\le j\le N_y}\int_{J_j}((\theta^+_{z,h} - \hat{\theta_{z,h}})\eta^+_{y,h})(x_{\frac{1}{2}}, y) \\
&+ \sum_{1\le i\le N_x} c_0\int_{I_i}[\eta^+_{x,h}(x, y_{\frac{1}{2}})]^2 + \sum_{1\le j\le N_y} c_0\int_{J_j}[\eta^+_{y,h}(x_{\frac{1}{2}}, y)]^2. \qquad (4.77)
\end{aligned}$$

We use the projection operators $\Pi_i, i = 1, 2, 3, 4$, their properties $(4.42) - -(4.53)$ and commutation property $\partial_t\Pi_i u = \Pi_i\partial_t u$. Then the right hand side is obtained

$$\begin{aligned}
&RHS = \\
&\int_\Omega\Big[\varepsilon_0(1+\chi^{(1)})[[\partial_t\eta_x]\eta_{x,h} + [\partial_t\eta_y]\eta_{y,h}] + \mu_0[\partial_t\theta_z]\theta_{z,h}\Big] \\
&+ \sum_{1\le i\le N_x}\Big[\int_{I_i}[-(\hat{\theta}_z\eta^-_{x,h})_{x,j+\frac{1}{2}} - (\hat{\theta}_z\eta^+_{x,h})_{x,j-\frac{1}{2}}] + \int_{K_{i,j}}\theta_z\partial_y\eta_{x,h}\Big] \\
&+ \sum_{1\le j\le N_y}\Big[\int_{J_j}[(\hat{\theta}_z\eta^-_{y,h})_{i+\frac{1}{2},y} - (\hat{\theta}_z\eta^+_{y,h})_{j-\frac{1}{2},y}]dy - \int_{K_{i,j}}\theta_z\partial_x\eta_{y,h}\Big] \\
&- \int_{K_{i,j}}\eta_y\partial_x\theta_{z,h} + \int_{K_{i,j}}\eta_{y,h}\partial_x\theta_{z,h} + \int_{K_{i,j}}\eta_x\partial_x\theta_{z,h} - \int_{K_{i,j}}\eta_{x,h}\partial_x\theta_{z,h} \\
&+ \varepsilon_0\chi^{(3)}\int_\Omega\Big[[\left(E^2_x + E^2_y\right) - \left(E^2_{x,h} + E^2_{y,h}\right)]\Big[[\partial_t E_x]\eta_{x,h} + [\partial_t E_y]\eta_{y,h}\Big] \\
&+ 2[E^2_x - E^2_{x,h}][\partial_t E_x]\eta_{x,h} + 2[E^2_y - E^2_{y,h}][\partial_t E_y]\eta_{y,h} + 2[E_xE_y - E_{x,h}E_{y,h}] \\
&\Big[[\partial_t E_y]\eta_{x,h} + [\partial_t E_x]\eta_{y,h}\Big] + 2E^2_{x,h}[\partial_t\eta_x]\eta_{x,h} + 2E^2_{y,h}[\partial_t\eta_y]\eta_{y,h} \\
&+ 2E_{x,h}E_{y,h}\Big[[\partial_t\eta_y]\eta_{x,h} + [\partial_t\eta_x]\eta_{y,h}\Big] + \left(E^2_{x,h} + E^2_{y,h}\right)\Big[[\partial_t\eta_x]\eta_{x,h} + [\partial_t\eta_y]\eta_{y,h}\Big]\Big].
\end{aligned}$$
(4.78)

Now we rewrite the nonlinear terms from the left hand side as

$$
\begin{aligned}
&\big(E_{x,h}^2+E_{y,h}^2\big)\frac{1}{2}\frac{d}{dt}\Big[\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}+\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\Big]\\
&+2\Big(E_{x,h}^2\frac{1}{2}\frac{d}{dt}\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}+E_{y,h}^2\frac{1}{2}\frac{d}{dt}\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\Big)\\
&+2\varepsilon_0\chi^{(3)}E_{x,h}E_{y,h}[\partial_t\eta_{y,h}]\eta_{x,h}+2E_{x,h}E_{y,h}[\partial_t\eta_{x,h}]\eta_{y,h}\\
&=\frac{1}{2}\frac{d}{dt}\Big[\big(E_{x,h}^2+E_{y,h}^2\big)\big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}+\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\big)\Big]\\
&-\Big[\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}+\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\Big]\frac{1}{2}\frac{d}{dt}\big(E_{x,h}^2+E_{y,h}^2\big)\\
&+\frac{d}{dt}\Big(E_{x,h}^2\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}\Big)+\frac{d}{dt}\Big(E_{y,h}^2\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\Big)\\
&-\Big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}\frac{d}{dt}E_{x,h}^2+\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\frac{d}{dt}E_{y,h}^2\Big)\\
&+2\frac{d}{dt}\Big[\varepsilon_0\chi^{(3)}E_{x,h}E_{y,h}\eta_{y,h}\eta_{x,h}\Big]-2[\eta_{x,h}\eta_{y,h}]\frac{d}{dt}[E_{x,h}E_{y,h}]\\
=&\frac{1}{2}\frac{d}{dt}\Big[\big(E_{x,h}^2+E_{y,h}^2\big)\big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}+\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\big)\Big]\\
&+\varepsilon_0\chi^{(3)}\frac{d}{dt}\Big(E_{x,h}\eta_{x,h}+E_{y,h}\eta_{y,h}\Big)^2\\
&-\Big[\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}+\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\Big]\frac{1}{2}\frac{d}{dt}\big(E_{x,h}^2+E_{y,h}^2\big)\\
&-\Big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}\frac{d}{dt}E_{x,h}^2+\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\frac{d}{dt}E_{y,h}^2\Big)\\
&-2[\eta_{x,h}\eta_{y,h}]\frac{d}{dt}[E_{x,h}E_{y,h}].
\end{aligned}
\tag{4.79}
$$

Next, using $\partial_t E_x=\partial_t\eta_x+\partial_t\Pi_1 E_x$ and $\partial_t E_y=\partial_t\eta_y+\partial_t\Pi_2 E_y$ in the nonlinear terms at the right hand side, we obtain

$$
\begin{aligned}
&\varepsilon_0\chi^{(3)}\int_\Omega\Big[[\big(E_x^2+E_y^2\big)-\big(E_{x,h}^2+E_{y,h}^2\big)]\Big[[\partial_t E_x]\eta_{x,h}+[\partial_t E_y]\eta_{y,h}\Big]\\
&+2[E_x^2-E_{x,h}^2][\partial_t E_x]\eta_{x,h}+2[E_y^2-E_{y,h}^2][\partial_t E_y]\eta_{y,h}+2[E_xE_y-E_{x,h}E_{y,h}]\\
&\cdot\Big[[\partial_t E_y]\eta_{x,h}+[\partial_t E_x]\eta_{y,h}\Big]+2E_{x,h}^2[\partial_t\eta_x]\eta_{x,h}+2E_{y,h}^2[\partial_t\eta_y]\eta_{y,h}\\
&+2E_{x,h}E_{y,h}\Big[[\partial_t\eta_y]\eta_{x,h}+[\partial_t\eta_x]\eta_{y,h}\Big]+\big(E_{x,h}^2+E_{y,h}^2\big)\Big[[\partial_t\eta_x]\eta_{x,h}+[\partial_t\eta_y]\eta_{y,h}\Big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
=&\varepsilon_0\chi^{(3)}\int_\Omega \Big[[\big(E_x^2+E_y^2\big)-\big(E_{x,h}^2+E_{y,h}^2\big)]\Big[[\partial_t\eta_x+\partial_t\Pi_1E_x]\eta_{x,h}+[\partial_t\eta_y\\
&+\partial_t\Pi_2E_y]\eta_{y,h}\Big]+\big(E_{x,h}^2+E_{y,h}^2\big)\Big[[\partial_t\eta_x]\eta_{x,h}+[\partial_t\eta_y]\eta_{y,h}\Big]+2[E_x^2-E_{x,h}^2][\partial_t\eta_x\\
&+\partial_t\Pi_1E_x]\eta_{x,h}+2E_{x,h}^2[\partial_t\eta_x]\eta_{x,h}+2[E_y^2-E_{y,h}^2][\partial_t\eta_y+\partial_t\Pi_2E_y]\eta_{y,h}\\
&+2E_{y,h}^2[\partial_t\eta_y]\eta_{y,h}+2[E_xE_y-E_{x,h}E_{y,h}]\Big[[\partial_t\eta_y+\partial_t\Pi_2E_y]\eta_{x,h}+[\partial_t\eta_x\\
&+\partial_t\Pi_1E_x]\eta_{y,h}\Big]+2E_{x,h}E_{y,h}\Big[[\partial_t\eta_y]\eta_{x,h}+[\partial_t\eta_x]\eta_{y,h}\Big]\Big]\\
=&\varepsilon_0\chi^{(3)}\int_\Omega \Big[[\big(E_x^2+E_y^2\big)-\big(E_{x,h}^2+E_{y,h}^2\big)]\Big[[\partial_t\Pi_1E_x]\eta_{x,h}+[\partial_t\Pi_2E_y]\eta_{y,h}\Big]\\
&+\big(E_x^2+E_y^2\big)\Big[[\partial_t\eta_x]\eta_{x,h}+[\partial_t\eta_y]\eta_{y,h}\Big]+2[E_x^2-E_{x,h}^2][\partial_t\Pi_1E_x]\eta_{x,h}\\
&+2E_x^2[\partial_t\eta_x]\eta_{x,h}+2[E_y^2-E_{y,h}^2][\partial_t\Pi_2E_y]\eta_{y,h}+2E_y^2[\partial_t\eta_y]\eta_{y,h}\\
&+2[E_xE_y-E_yE_{x,h}+E_yE_{x,h}-E_{x,h}E_{y,h}]\Big[[\partial_t\Pi_2E_y]\eta_{x,h}+[\partial_t\Pi_1E_x]\eta_{y,h}\Big]\\
&+2E_xE_y\Big[[\partial_t\eta_y]\eta_{x,h}+[\partial_t\eta_x]\eta_{y,h}\Big]\Big]\\
=&\varepsilon_0\chi^{(3)}\int_\Omega \Big[[\big(E_x+E_{x,h}\big)\big(E_x-E_{x,h}\big)+\big(E_y+E_{y,h}\big)\big(E_y-E_{y,h}\big)]\Big[[\partial_t\Pi_1E_x]\eta_{x,h}\\
&+[\partial_t\Pi_2E_y]\eta_{y,h}\Big]+\big(E_x^2+E_y^2\big)\Big[[\partial_t\eta_x]\eta_{x,h}+[\partial_t\eta_y]\eta_{y,h}\Big]+2[\big(E_x+E_{x,h}\big)\big(E_x\\
&-E_{x,h}\big)][\partial_t\Pi_1E_x]\eta_{x,h}+2E_x^2[\partial_t\eta_x]\eta_{x,h}+2[\big(E_y+E_{y,h}\big)\big(E_y-E_{y,h}\big)][\partial_t\Pi_2E_y]\eta_{y,h}\\
&+2E_y^2[\partial_t\eta_y]\eta_{y,h}+2[E_y\big(E_x-E_{x,h}\big)+E_{x,h}\big(E_y-E_{y,h}\big)]\Big[[\partial_t\Pi_2E_y]\eta_{x,h}\\
&+[\partial_t\Pi_1E_x]\eta_{y,h}\Big]+2E_xE_y\Big[[\partial_t\eta_y]\eta_{x,h}+[\partial_t\eta_x]\eta_{y,h}\Big]\Big]. \qquad (4.80)
\end{aligned}
$$

Furthermore, since $E_x-E_{x,h}=\eta_x-\eta_{x,h}$ and $E_y-E_{y,h}=\eta_y-\eta_{y,h}$, we have

$$
\begin{aligned}
&\varepsilon_0\chi^{(3)}\int_\Omega \Big[[\big(E_x+E_{x,h}\big)\big(\eta_x-\eta_{x,h}\big)+\big(E_y+E_{y,h}\big)\big(\eta_y-\eta_{y,h}\big)]\Big[[\partial_t\Pi_1E_x]\eta_{x,h}\\
&+[\partial_t\Pi_2E_y]\eta_{y,h}\Big]+\big(E_x^2+E_y^2\big)\Big[[\partial_t\eta_x]\eta_{x,h}+[\partial_t\eta_y]\eta_{y,h}\Big]+2[\big(E_x+E_{x,h}\big)\big(\eta_x\\
&-\eta_{x,h}\big)][\partial_t\Pi_1E_x]\eta_{x,h}+2E_x^2[\partial_t\eta_x]\eta_{x,h}+2[\big(E_y+E_{y,h}\big)\big(\eta_y-\eta_{y,h}\big)][\partial_t\Pi_2E_y]\eta_{y,h}\\
&+2E_y^2[\partial_t\eta_y]\eta_{y,h}+2[E_y\big(\eta_x-\eta_{x,h}\big)+E_{x,h}\big(\eta_y-\eta_{y,h}\big)]\Big[[\partial_t\Pi_2E_y]\eta_{x,h}\\
&+[\partial_t\Pi_1E_x]\eta_{y,h}\Big]+2E_xE_y\Big[[\partial_t\eta_y]\eta_{x,h}+[\partial_t\eta_x]\eta_{y,h}\Big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
=&\varepsilon_0\chi^{(3)}\int_\Omega \Big[[\big(E_x\eta_x - E_x\eta_{x,h} + E_{x,h}\eta_x - E_{x,h}\eta_{x,h}\big)\\
&+\big(E_y\eta_y - E_y\eta_{y,h} + E_{y,h}\eta_y - E_{y,h}\eta_{y,h}\big)]\Big[[\partial_t\Pi_1E_x]\eta_{x,h} + [\partial_t\Pi_2E_y]\eta_{y,h}\Big]\\
&+\big(E_x^2+E_y^2\big)\Big[[\partial_t\eta_x]\eta_{x,h} + [\partial_t\eta_y]\eta_{y,h}\Big] + 2[\big(E_x\eta_x - E_x\eta_{x,h} + E_{x,h}\eta_x\\
&- E_{x,h}\eta_{x,h}\big)][\partial_t\Pi_1E_x]\eta_{x,h} + 2E_x^2[\partial_t\eta_x]\eta_{x,h} + 2[\big(E_y\eta_y - E_y\eta_{y,h} + E_{y,h}\eta_y\\
&- E_{y,h}\eta_{y,h}\big)][\partial_t\Pi_2E_y]\eta_{y,h} + 2E_y^2[\partial_t\eta_y]\eta_{y,h} + 2E_y\big(\eta_x-\eta_{x,h}\big)[\partial_t\Pi_2E_y]\eta_{x,h}\\
&+ 2E_y\big(\eta_x-\eta_{x,h}\big)[\partial_t\Pi_1E_x]\eta_{y,h} + 2E_{x,h}\big(\eta_y-\eta_{y,h}\big)[\partial_t\Pi_2E_y]\eta_{x,h}\\
&+ 2E_{x,h}\big(\eta_y\eta_{y,h}-\eta_{y,h}\big)[\partial_t\Pi_1E_x] + 2E_xE_y\Big[[\partial_t\eta_y]\eta_{x,h} + [\partial_t\eta_x]\eta_{y,h}\Big]\Big]\\
=&\varepsilon_0\chi^{(3)}\int_\Omega \Big[E_x\eta_x[\partial_t\Pi_1E_x]\eta_{x,h} - E_x[\partial_t\Pi_1E_x]\eta_{x,h}\eta_{x,h} + E_{x,h}[\partial_t\Pi_1E_x]\eta_{x,h}\eta_x\\
&- E_{x,h}[\partial_t\Pi_1E_x]\eta_{x,h}\eta_{x,h} + E_y[\partial_t\Pi_1E_x]\eta_{x,h}\eta_y - E_y[\partial_t\Pi_1E_x]\eta_{x,h}\eta_{y,h}\\
&+ E_{y,h}\eta_y[\partial_t\Pi_1E_x]\eta_{x,h} - E_{y,h}\eta_{y,h}[\partial_t\Pi_1E_x]\eta_{x,h}\\
&+ \big[E_x\eta_x[\partial_t\Pi_2E_y]\eta_{y,h} - E_x\eta_{x,h}[\partial_t\Pi_2E_y]\eta_{y,h} + E_{x,h}\eta_x[\partial_t\Pi_2E_y]\eta_{y,h}\\
&- E_{x,h}\eta_{x,h}[\partial_t\Pi_2E_y]\eta_{y,h} + E_y\eta_y[\partial_t\Pi_2E_y]\eta_{y,h} - E_y\eta_{y,h}[\partial_t\Pi_2E_y]\eta_{y,h}\\
&+ E_{y,h}\eta_y[\partial_t\Pi_2E_y]\eta_{y,h} - E_{y,h}\eta_{y,h}[\partial_t\Pi_2E_y]\eta_{y,h}\big] + \big(E_x^2+E_y^2\big)[\partial_t\eta_x]\eta_{x,h}\\
&+ \big(E_x^2+E_y^2\big)[\partial_t\eta_y]\eta_{y,h} + 2E_x\eta_x[\partial_t\Pi_1E_x]\eta_{x,h} - 2E_x\eta_{x,h}[\partial_t\Pi_1E_x]\eta_{x,h}\\
&+ 2E_{x,h}\eta_x[\partial_t\Pi_1E_x]\eta_{x,h} - 2E_{x,h}\eta_{x,h}[\partial_t\Pi_1E_x]\eta_{x,h} + 2E_x^2[\partial_t\eta_x]\eta_{x,h}]\\
&+ 2E_y\eta_y[\partial_t\Pi_2E_y]\eta_{y,h} - 2E_y\eta_{y,h}[\partial_t\Pi_2E_y]\eta_{y,h} + 2E_{y,h}\eta_y[\partial_t\Pi_2E_y]\eta_{y,h}\\
&- 2E_{y,h}\eta_{y,h}[\partial_t\Pi_2E_y]\eta_{y,h} + 2E_y^2[\partial_t\eta_y]\eta_{y,h} + 2E_y\eta_x[\partial_t\Pi_2E_y]\eta_{x,h}\\
&- 2E_y\eta_{x,h}[\partial_t\Pi_2E_y]\eta_{x,h} + 2E_y\eta_x[\partial_t\Pi_1E_x]\eta_{y,h} - 2E_y\eta_{x,h}[\partial_t\Pi_1E_x]\eta_{y,h}\\
&+ 2E_{x,h}\eta_y[\partial_t\Pi_2E_y]\eta_{x,h} - 2E_{x,h}\eta_{y,h}[\partial_t\Pi_2E_y]\eta_{x,h} + 2E_{x,h}\eta_y[\partial_t\Pi_1E_x]\eta_{y,h}\\
&- 2E_{x,h}\eta_{y,h}[\partial_t\Pi_1E_x]\eta_{y,h} + 2E_xE_y[\partial_t\eta_y]\eta_{x,h} + 2E_xE_y[\partial_t\eta_x]\eta_{y,h}\Big].
\end{aligned}
$$

Further we have

$$
\begin{aligned}
&\varepsilon_0\chi^{(3)}\int_\Omega \Bigg[\Big[E_x\eta_x[\partial_t\Pi_1E_x] + E_{x,h}\eta_x[\partial_t\Pi_1E_x] + E_y\eta_y[\partial_t\Pi_1E_x] \\
&+ E_{y,h}\eta_y[\partial_t\Pi_1E_x] + \big(E_x^2+E_y^2\big)[\partial_t\eta_x] + 2E_x\eta_x[\partial_t\Pi_1E_x] + 2E_{x,h}\eta_x[\partial_t\Pi_1E_x] \\
&+ 2E_x^2[\partial_t\eta_x] + 2E_y\eta_x[\partial_t\Pi_2E_y] + 2E_{x,h}\eta_y[\partial_t\Pi_2E_y] + 2E_xE_y[\partial_t\eta_y]\Big]\eta_{x,h} \\
&\\
&+ \Big[E_x\eta_x[\partial_t\Pi_2E_y] + E_{x,h}\eta_x[\partial_t\Pi_2E_y] + E_y\eta_y[\partial_t\Pi_2E_y] + E_{y,h}\eta_y[\partial_t\Pi_2E_y] \\
&+ \big(E_x^2+E_y^2\big)[\partial_t\eta_y] + 2E_y\eta_y[\partial_t\Pi_2E_y] + 2E_{y,h}\eta_y[\partial_t\Pi_2E_y] + 2E_y^2[\partial_t\eta_y] \\
&+ 2E_y\eta_x[\partial_t\Pi_1E_x] + 2E_{x,h}\eta_y[\partial_t\Pi_1E_x] + 2E_xE_y[\partial_t\eta_x]\Big]\eta_{y,h} \\
&\\
&+ \Big[-E_x[\partial_t\Pi_1E_x] - E_{x,h}[\partial_t\Pi_1E_x] - 2E_x[\partial_t\Pi_1E_x] - 2E_{x,h}[\partial_t\Pi_1E_x] \\
&- 2E_y[\partial_t\Pi_2E_y]\Big]\eta_{x,h}^2 \\
&\\
&+ \Big[-E_y[\partial_t\Pi_2E_y] - E_{y,h}[\partial_t\Pi_2E_y] - 2E_y[\partial_t\Pi_2E_y] - 2E_{y,h}[\partial_t\Pi_2E_y] \\
&- 2E_{x,h}[\partial_t\Pi_1E_x]\Big]\eta_{y,h}^2 \\
&\\
&+ \Big[-E_y[\partial_t\Pi_1E_x] - E_{y,h}[\partial_t\Pi_1E_x] - E_x[\partial_t\Pi_2E_y] - E_{x,h}[\partial_t\Pi_2E_y] \\
&- 2E_y[\partial_t\Pi_1E_x] - 2E_{x,h}[\partial_t\Pi_2E_y]\Big]\eta_{x,h}\eta_{y,h}\Bigg]. \qquad (4.81)
\end{aligned}
$$

Substituting the equation (4.79) into the equation (4.77), and shifting the last three terms to the RHS, we get

$$\begin{aligned}
LHS =& \frac{1}{2}\frac{d}{dt}[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_{z,h}\|^2_{\mu_0}] \\
&+ \frac{1}{2}\frac{d}{dt}\Big[\big(E^2_{x,h} + E^2_{y,h}\big)\big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}} + \|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\big)\Big] \\
&+ \varepsilon_0\chi^{(3)}\frac{d}{dt}\Big(E_{x,h}\eta_{x,h} + E_{y,h}\eta_{y,h}\Big)^2 + \sum_{1\le i\le N_x}\int_{I_i}((\hat{\theta_{z,h}} \\
&- \theta^+_{z,h})\eta^+_{x,h})(x, y_{\frac{1}{2}}) + \sum_{1\le j\le N_y}\int_{J_j}((\theta^+_{z,h} - \hat{\theta_{z,h}})\eta^+_{y,h})(x_{\frac{1}{2}}, y) \\
&+ \sum_{1\le i\le N_x} c_0\int_{I_i}[\eta^+_{x,h}(x, y_{\frac{1}{2}})]^2 + \sum_{1\le j\le N_y} c_0\int_{J_j}[\eta^+_{y,h}(x_{\frac{1}{2}}, y)]^2. \qquad (4.82)
\end{aligned}$$

Note that the following terms $\sum_{1\le i\le N_x}\int_{I_i}((\hat{\theta_{z,h}} - \theta^+_{z,h})\eta^+_{x,h})(x, y_{\frac{1}{2}}) + \sum_{1\le j\le N_y}\int_{J_j}((\theta^+_{z,h} - \hat{\theta_{z,h}})\eta^+_{y,h})(x_{\frac{1}{2}}, y)$ are expressed by the boundary fluxes (4.17) and (4.19). Then the left hand side reads as

$$\begin{aligned}
LHS =& \frac{1}{2}\frac{d}{dt}[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_{z,h}\|^2_{\mu_0}] \\
&+ \frac{1}{2}\frac{d}{dt}\Big[\big(E^2_{x,h} + E^2_{y,h}\big)\big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}} + \|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\big)\Big] \\
&+ \varepsilon_0\chi^{(3)}\frac{d}{dt}\Big(E_{x,h}\eta_{x,h} + E_{y,h}\eta_{y,h}\Big)^2 \\
&+ \sum_{1\le i\le N_x} c_0\int_{I_i}[\eta^+_{x,h}(x, y_{\frac{1}{2}})]^2 + \sum_{1\le j\le N_y} c_0\int_{J_j}[\eta^+_{y,h}(x_{\frac{1}{2}}, y)]^2. \qquad (4.83)
\end{aligned}$$

The right hand side is

$$
\begin{aligned}
&RHS = \\
&\int_\Omega \Big[\varepsilon_0(1+\chi^{(1)})[[\partial_t\eta_x]\eta_{x,h} + [\partial_t\eta_y]\eta_{y,h}] + \mu_0[\partial_t\theta_z]\theta_{z,h}\Big] \\
&+ \sum_{1\le i\le N_x} \Big[\int_{I_i} [-(\hat{\theta}_z\eta_{x,h}^-)_{x,j+\frac{1}{2}} - (\hat{\theta}_z\eta_{x,h}^+)_{x,j-\frac{1}{2}}] + \int_{K_{i,j}} \theta_z\partial_y\eta_{x,h}\Big] \\
&+ \sum_{1\le j\le N_y} \Big[\int_{J_j} [(\hat{\theta}_z\eta_{y,h}^-)_{i+\frac{1}{2},y} - (\hat{\theta}_z\eta_{y,h}^+)_{j-\frac{1}{2},y}]dy - \int_{K_{i,j}} \theta_z\partial_x\eta_{y,h}\Big] \\
&- \int_{K_{i,j}} \eta_y\partial_x\theta_{z,h} + \int_{K_{i,j}} \eta_{y,h}\partial_x\theta_{z,h} + \int_{K_{i,j}} \eta_x\partial_x\theta_{z,h} - \int_{K_{i,j}} \eta_{x,h}\partial_x\theta_{z,h} \\
&+ \varepsilon_0\chi^{(3)}\int_\Omega \Bigg[\Big[E_x\eta_x[\partial_t\Pi_1E_x] + E_{x,h}\eta_x[\partial_t\Pi_1E_x] + E_y\eta_y[\partial_t\Pi_1E_x] \\
&+ E_{y,h}\eta_y[\partial_t\Pi_1E_x] + \big(E_x^2+E_y^2\big)[\partial_t\eta_x] + 2E_x\eta_x[\partial_t\Pi_1E_x] + 2E_{x,h}\eta_x[\partial_t\Pi_1E_x] \\
&+ 2E_x^2[\partial_t\eta_x] + 2E_y\eta_x[\partial_t\Pi_2E_y] + 2E_{x,h}\eta_y[\partial_t\Pi_2E_y] + 2E_xE_y[\partial_t\eta_y]\Big]\eta_{x,h} \\
&+ \Big[E_x\eta_x[\partial_t\Pi_2E_y] + E_{x,h}\eta_x[\partial_t\Pi_2E_y] + E_y\eta_y[\partial_t\Pi_2E_y] + E_{y,h}\eta_y[\partial_t\Pi_2E_y] \\
&+ \big(E_x^2+E_y^2\big)[\partial_t\eta_y] + 2E_y\eta_y[\partial_t\Pi_2E_y] + 2E_{y,h}\eta_y[\partial_t\Pi_2E_y] + 2E_y^2[\partial_t\eta_y] \\
&+ 2E_y\eta_x[\partial_t\Pi_1E_x] + 2E_{x,h}\eta_y[\partial_t\Pi_1E_x] + 2E_xE_y[\partial_t\eta_x]\Big]\eta_{y,h} \\
&+ \Big[-E_x[\partial_t\Pi_1E_x] - E_{x,h}[\partial_t\Pi_1E_x] - 2E_x[\partial_t\Pi_1E_x] - 2E_{x,h}[\partial_t\Pi_1E_x] \\
&- 2E_y[\partial_t\Pi_2E_y]\Big]\eta_{x,h}^2 \\
&+ \Big[-E_y[\partial_t\Pi_2E_y] - E_{y,h}[\partial_t\Pi_2E_y] - 2E_y[\partial_t\Pi_2E_y] - 2E_{y,h}[\partial_t\Pi_2E_y] \\
&- 2E_{x,h}[\partial_t\Pi_1E_x]\Big]\eta_{y,h}^2 \\
&+ \Big[-E_y[\partial_t\Pi_1E_x] - E_{y,h}[\partial_t\Pi_1E_x] - E_x[\partial_t\Pi_2E_y] - E_{x,h}[\partial_t\Pi_2E_y] \\
&- 2E_y[\partial_t\Pi_1E_x] - 2E_{x,h}[\partial_t\Pi_2E_y]\Big]\eta_{x,h}\eta_{y,h}\Bigg] \\
&+ \Big[\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}} + \|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\Big]\frac{1}{2}\frac{d}{dt}\big(E_{x,h}^2+E_{y,h}^2\big) \\
&+ \Big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}\frac{d}{dt}E_{x,h}^2 + \|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\frac{d}{dt}E_{y,h}^2\Big) \\
&+ 2[\eta_{x,h}\eta_{y,h}]\frac{d}{dt}[E_{x,h}E_{y,h}]. \qquad (4.84)
\end{aligned}
$$

The linear terms from the right hand side of equation (4.84) are figured out by employing the inequality (3) from Lemma 5.1. This gives

$$
\begin{aligned}
&\int_\Omega \Big[\varepsilon_0(1+\chi^{(1)})[[\partial_t \eta_x]\eta_{x,h} + [\partial_t \eta_y]\eta_{y,h}] + \mu_0[\partial_t \theta_z]\theta_{z,h}\Big] \\
&\le \|(1+\chi^{(1)})\|_{L^\infty(\Omega)}\Big[\|\partial_t \eta_x\|_{\varepsilon_0}\|\eta_{x,h}\|_{\varepsilon_0} + \|\partial_t \eta_y\|_{\varepsilon_0}\|\eta_{y,h}\|_{\varepsilon_0}\Big] + \|\partial_t \theta_z\|_{\mu_0}\|\theta_{z,h}\|_{\mu_0} \\
&\le C_{i11}h^k[\|\eta_{x,h}\|_{\varepsilon_0} + \|\eta_{y,h}\|_{\varepsilon_0} + \|\theta_{z,h}\|_{\mu_0}]. \qquad (4.85)
\end{aligned}
$$

Applying Lemmas 4.7–4.8 to the terms below from the right hand side, we obtain

$$
\begin{aligned}
&+\sum_{1\le i\le N_x}\Big[\int_{I_i}[-(\hat{\theta}_z\eta^-_{x,h})_{x,j+\frac{1}{2}} - (\hat{\theta}_z\eta^+_{x,h})_{x,j-\frac{1}{2}}] + \int_{K_{i,j}}\theta_z\partial_y\eta_{x,h}\Big] \\
&-\sum_{1\le i\le N_x} c_0\int_{I_i}[\eta^+_{x,h}(x,y_{\frac{1}{2}})]^2 \le C_{i12}h^{2k+2} + \|\eta_{x,h}\|^2, \qquad (4.86) \\
&+\sum_{1\le j\le N_y}\Big[\int_{J_j}[(\hat{\theta}_z\eta^-_{y,h})_{i+\frac{1}{2},y} - (\hat{\theta}_z\eta^+_{y,h})_{j-\frac{1}{2},y}]dy - \int_{K_{i,j}}\theta_z\partial_x\eta_{y,h}\Big] \\
&-\sum_{1\le j\le N_y} c_0\int_{J_j}[\eta^+_{y,h}(x_{\frac{1}{2}},y)]^2 \le C_{i13}h^{2k+2} + \|\eta_{y,h}\|^2. \qquad (4.87)
\end{aligned}
$$

Next the nonlinear terms from the right hand side are simplified as

$$
\begin{aligned}
&\|\chi^{(3)}\|_{L^\infty(\Omega)}\Bigg[\Big[\|E_x\|_{L^\infty(\Omega)}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))}\|\eta_x\|_{\varepsilon_0} \\
&+\|E_{x,h}\|_{L^\infty(\Omega)}\|\eta_x\|_{\varepsilon_0}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))} \\
&+\|E_y\|_{L^\infty(\Omega)}\|\eta_y\|\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))} + \|E_{y,h}\|_{L^\infty(\Omega)}\|\eta_y\| \\
&\cdot\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))} + \|E_x^2 + E_y^2\|_{L^\infty(\Omega)}\|\partial_t\eta_x\|_{\varepsilon_0} + 2\|E_x\|_{L^\infty(\Omega)}\|\eta_x\|_{\varepsilon_0} \\
&\cdot\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))} + 2\|E_{x,h}\|_{L^\infty(\Omega)}\|\eta_x\|_{\varepsilon_0}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))} \\
&+2\|E_x^2\|_{L^\infty(\Omega)}\|\partial_t\eta_x\|_{\varepsilon_0} + \|2E_y\|_{L^\infty(\Omega)}\|\eta_x\|_{\varepsilon_0}\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))} \\
&+2\|E_{x,h}\|_{L^\infty(\Omega)}\|\eta_y\|\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))} \\
&+2\|E_x\|_{L^\infty(\Omega)}\|E_y\|_{L^\infty(\Omega)}\|\partial_t\eta_y\|\Big]\|\eta_{x,h}\|_{\varepsilon_0}
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big[\|E_x\|_{L^\infty(\Omega)}\|\eta_x\|_{\varepsilon_0}\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))} + \|E_{x,h}\|_{L^\infty(\Omega)}\|\eta_x\|_{\varepsilon_0}\\
&\cdot \|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))} + \|E_y\|_{L^\infty(\Omega)}\|\eta_{y\|}\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))}+\\
&\|E_{y,h}\|_{L^\infty(\Omega)}\|\eta_y\|\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))} + \|E_x^2 + E_y^2\|_{L^\infty(\Omega)}\|\partial_t\eta_y\|\\
&+ 2\|E_y\|_{L^\infty(\Omega)}\|\eta_y\|\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))}\\
&+ 2\|E_{y,h}\|_{L^\infty(\Omega)}\|\eta_y\|\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))} + \|2E_y^2\|_{L^\infty(\Omega)}\|\partial_t\eta_y\|\\
&+ \|E_y\|_{L^\infty(\Omega)}\|\eta_x\|\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))} + 2\|E_{x,h}\|_{L^\infty(\Omega)}\|\eta_y\|\\
&\cdot \|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))}\\
&+ 2\|E_x\|_{L^\infty(\Omega)}\|E_y\|_{L^\infty(\Omega)}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))}\Big]\|\eta_{y,h}\|_{\varepsilon_0}\\
&+ \Big[\|E_x\|_{L^\infty(\Omega)}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))} + \|E_{x,h}\|_{L^\infty(\Omega)}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))}\\
&+ \|E_x\|_{L^\infty(\Omega)}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))} + 2\|E_{x,h}\|_{L^\infty(\Omega)}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))}\\
&+ 2\|E_y\|_{L^\infty(\Omega)}\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))}\Big]\|\eta_{x,h}\|^2\\
&+ \Big[\|E_y\|_{L^\infty(\Omega)}\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))} + \|E_{y,h}\|_{L^\infty(\Omega)}\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))}\\
&+ 2\|E_y\|_{L^\infty(\Omega)}\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))} + 2\|E_{y,h}\|_{L^\infty(\Omega)}\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))}\\
&+ 2\|E_{x,h}\|_{L^\infty(\Omega)}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))}\Big]\|\eta_{y,h}\|^2\\
&+ \Big[\|E_y\|_{L^\infty(\Omega)}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))} + \|E_{y,h}\|_{L^\infty(\Omega)}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))}\\
&+ \|E_x\|_{L^\infty(\Omega)}\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))} + \|E_{x,h}\|_{L^\infty(\Omega)}\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))}\\
&+ 2\|E_y\|_{L^\infty(\Omega)}\|\partial_t\Pi_1 E_x\|_{C(0,T,L^\infty(\Omega))} + 2\|E_{x,h}\|_{L^\infty(\Omega)}\|\partial_t\Pi_2 E_y\|_{C(0,T,L^\infty(\Omega))}\Big]\\
&\cdot \|\eta_{x,h}\|\|\eta_{y,h}\|\Bigg]\\
&\le C_{i6}h^k[\|\eta_{x,h}\|_{\varepsilon_0} + \|\eta_{y,h}\|_{\mu_0}] + C_{i7}\|\eta_{x,h}\|^2 + C_{i8}\|\eta_{y,h}\|^2 + 2C_{i9}\|\eta_{x,h}\|\|\eta_{y,h}\|\\
&\le C_{i14}h^k[\|\eta_{x,h}\|_{\varepsilon_0} + \|\eta_{y,h}\|_{\varepsilon_0}] + C_{i15}\Big(\|\eta_{x,h}\|_{\varepsilon_0} + \|\eta_{y,h}\|_{\varepsilon_0}\Big)^2. \qquad (4.88)
\end{aligned}
$$

Furthermore the next nonlinear terms are simplified as

$$
\begin{aligned}
&\Big[\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}} + \|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\Big]\frac{1}{2}\partial_t\big(E^2_{x,h} + E^2_{y,h}\big) \\
&+ \Big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}\partial_t E^2_{x,h} + \|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\partial_t E^2_{y,h}\Big) + 2[\eta_{x,h}\eta_{y,h}]\partial_t[E_{x,h}E_{y,h}] \\
&\le \Big[\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\eta_{x,h}\|^2_{\varepsilon_0} + \|\chi^{(3)}\|_{L^\infty(\Omega)}\|\eta_{y,h}\|^2_{\varepsilon_0}\Big]\|\frac{1}{2}\partial_t\big(E^2_{x,h} + E^2_{y,h}\big)\|_{C(0,T,L^\infty(\Omega))} \\
&+ \Big(\|\eta_{x,h}\|^2_{\varepsilon_0}\|\chi^{(3)}\|_{L^\infty(\Omega)}\|\partial_t E^2_{x,h}\|_{C(0,T,L^\infty(\Omega))} + \|\eta_{y,h}\|^2_{\varepsilon_0}\|\chi^{(3)}\|_{L^\infty(\Omega)} \\
&\cdot \|\partial_t E^2_{y,h}\|_{C(0,T,L^\infty(\Omega))}\Big) + 2\|\eta_{x,h}\|\eta_{y,h}\|\|\partial_t[E_{x,h}E_{y,h}]\|_{C(0,T,L^\infty(\Omega))} \\
&\le \Big[\|\eta_{x,h}\|^2_{\varepsilon_0} + \|\eta_{y,h}\|^2_{\varepsilon_0}\Big]C_{i1} + \Big(\|\eta_{x,h}\|^2_{\varepsilon_0}C_{i2} + \|\eta_{y,h}\|^2_{\varepsilon_0}C_{i3}\Big) + 2\|\eta_{x,h}\|\eta_{y,h}\|C_{i4} \\
&\le \Big(\|\eta_{x,h}\|^2_{\varepsilon_0}C_{j1} + \|\eta_{y,h}\|^2_{\varepsilon_0}C_{j2}\Big) + 2\|\eta_{x,h}\|\eta_{y,h}\|C_{j3} \\
&\le C_{i16}\Big(\|\eta_{x,h}\|_{\varepsilon_0} + \|\eta_{y,h}\|_{\varepsilon_0}\Big)^2. \qquad (4.89)
\end{aligned}
$$

Adding all the right hand side estimates (4.85)–(4.89), combining then with left the hand side (4.83), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_{z,h}\|^2_{\mu_0}] \\
&+ \frac{1}{2}\frac{d}{dt}\Big[\big(E^2_{x,h} + E^2_{y,h}\big)\big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}} + \|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\big)\Big] \\
&+ \varepsilon_0\chi^{(3)}\frac{d}{dt}\Big(E_{x,h}\eta_{x,h} + E_{y,h}\eta_{y,h}\Big)^2 \\
&\le C_{i11}h^k[\|\eta_{x,h}\|_{\varepsilon_0} + \|\eta_{y,h}\|_{\varepsilon_0} + \|\theta_{z,h}\|_{\mu_0}] + C_{i12}h^{2k+2} + \|\eta_{x,h}\|^2 \\
&+ C_{i13}h^{2k+2} + \|\eta_{y,h}\|^2 + C_{i14}h^k[\|\eta_{x,h}\|_{\varepsilon_0} + \|\eta_{y,h}\|_{\varepsilon_0}] + C_{i15}\Big(\|\eta_{x,h}\|_{\varepsilon_0} + \|\eta_{y,h}\|_{\varepsilon_0}\Big)^2 \\
&+ C_{i16}\Big(\|\eta_{x,h}\|_{\varepsilon_0} + \|\eta_{y,h}\|_{\varepsilon_0}\Big)^2 \\
&\le C_{i17}h^k[\|\eta_{x,h}\|_{\varepsilon_0} + \|\eta_{y,h}\|_{\varepsilon_0} + \|\theta_{z,h}\|_{\mu_0}] + C_{i18}h^{2k+2} + C_{i19}\Big[\|\eta_{x,h}\|^2_{\varepsilon_0} + \|\eta_{y,h}\|^2_{\varepsilon_0}\Big].
\end{aligned}
$$

Setting

$$k_h(t) := \sqrt{\|\eta_{x,h}\|^2_{\varepsilon_0} + \|\eta_{y,h}\|^2_{\varepsilon_0} + \|\theta_{z,h}\|^2_{\mu_0}},$$

we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\theta_{z,h}\|^2_{\mu_0}] \\
&+\frac{1}{2}\frac{d}{dt}\Big[\big(E^2_{x,h}+E^2_{y,h}\big)\big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}+\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\big)\Big] \\
&+\varepsilon_0\chi^{(3)}\frac{d}{dt}\Big(E_{x,h}\eta_{x,h}+E_{y,h}\eta_{y,h}\Big)^2 \\
&\le C_{i17}h^k\sqrt{2}k_h(t)+C_{i18}h^{2k+2}+2C_{i19}\Big[\|\eta_{x,h}\|^2_{\varepsilon_0}+\|\eta_{y,h}\|^2_{\varepsilon_0}\Big] \\
&\le C_{i17}h^k\sqrt{2}k_h(t)+C_{i18}h^{2k+2}+C_{i19}k^2_h(t).
\end{aligned}$$

Integrating this inequality with respect to time, we obtain

$$\begin{aligned}
&\frac{1}{2}\Big[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\theta_{z,h}\|^2_{\mu_0}\Big] \\
&+\frac{1}{2}\big(E^2_{x,h}+E^2_{y,h}\big)\big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}+\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\big) \\
&+\varepsilon_0\chi^{(3)}\Big(E_{x,h}\eta_{x,h}+E_{y,h}\eta_{y,h}\Big)^2 \\
&\le\frac{1}{2}\Big[\|\eta^0_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\eta^0_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\theta^0_{z,h}\|^2_{\mu_0}\Big] \\
&+\big((E^0_{x,h})^2+(E^0_{y,h})^2\big)\big(\|\eta^0_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}+\|\eta^0_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\big) \\
&+\varepsilon_0\chi^{(3)}\Big(E^0_{x,h}\eta^0_{x,h}+E^0_{y,h}\eta^0_{y,h}\Big)^2 \\
&+\int_0^t\Big[C_{i17}h^k\sqrt{2}k_h(s)+C_{i18}h^{2k+2}+C_{i19}k^2_h(s)\Big]\,ds. \qquad (4.90)
\end{aligned}$$

By the monotonicity of the weighted norms w.r.t. the weight and the non-negativity of the integral term on the left-hand side, we see that

$$\begin{aligned}
\frac{1}{2}k^2_h(t)\le{}&\frac{1}{2}\Big[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\theta_{z,h}\|^2_{\mu_0}\Big] \\
&+\frac{1}{2}\big(E^2_{x,h}+E^2_{y,h}\big)\big(\|\eta_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}+\|\eta_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\big) \\
&+\varepsilon_0\chi^{(3)}\Big(E_{x,h}\eta_{x,h}+E_{y,h}\eta_{y,h}\Big)^2. \qquad (4.91)
\end{aligned}$$

On the other hand, we have the estimates

$$\|\eta^0_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\le\|1+\chi^{(1)}\|_{L^\infty(\Omega)}\|\eta^0_{x,h}\|^2_{\varepsilon_0}\le\|1+\chi^{(1)}\|_{L^\infty(\Omega)}w^2_h(0), \qquad (4.92)$$

$$\|\eta^0_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\le\|1+\chi^{(1)}\|_{L^\infty(\Omega)}\|\eta^0_{y,h}\|^2_{\varepsilon_0}\le\|1+\chi^{(1)}\|_{L^\infty(\Omega)}w^2_h(0) \qquad (4.93)$$

and

$$\frac{1}{2}\big((E^0_{x,h})^2+(E^0_{y,h})^2\big)\Big[\|\eta^0_{x,h}\|^2_{\varepsilon_0\chi^{(3)}}+\|\eta^0_{y,h}\|^2_{\varepsilon_0\chi^{(3)}}\Big]+\varepsilon_0\chi^{(3)}\Big(E^0_{x,h}\eta^0_{x,h}+E^0_{y,h}\eta^0_{y,h}\Big)^2$$
$$\leq \frac{1}{2}\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(E^0_{x,h})^2+(E^0_{y,h})^2\|_{L^\infty(\Omega)}\Big[\|\eta^0_{x,h}\|^2_{\varepsilon_0}+\|\eta^0_{y,h}\|^2_{\varepsilon_0}\Big]$$
$$+2\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(E^0_{x,h})^2+(E^0_{y,h})^2\|_{L^\infty(\Omega)}\Big[\|\eta^0_{x,h}\|^2_{\varepsilon_0}+\|\eta^0_{y,h}\|^2_{\varepsilon_0}\Big]$$
$$\leq \frac{5}{2}\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(E^0_{x,h})^2+(E^0_{y,h})^2\|_{L^\infty(\Omega)}k^2_h(0). \tag{4.94}$$

Combining (4.90), (4.91), (4.92),(4.93) with (4.94), we get

$$\frac{1}{2}k^2_h(t)\leq \|1+\chi^{(1)}\|_{L^\infty(\Omega)}k^2_h(0)+\frac{5}{2}\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(E^0_{x,h})^2+(E^0_{y,h})^2\|_{L^\infty(\Omega)}w^2_h(0)$$
$$+\int_0^t\Big[C_{i17}h^k\sqrt{2}k_h(s)+C_{i18}h^{2k+2}+C_{i19}k^2_h(s)\Big]\,ds,$$

or, equivalently,

$$k^2_h(t)\leq C^2k^2_h(0)+\int_0^t\Big[C_{i17}h^k\sqrt{2}k_h(s)+C_{i18}h^{2k+2}+C_{i19}k^2_h(s)\Big]\,ds, \tag{4.95}$$

where $C^2:=\|1+\chi^{(1)}\|_{L^\infty(\Omega)}+\frac{5}{2}\|\chi^{(3)}\|_{L^\infty(\Omega)}\|(E^0_{x,h})^2+(E^0_{y,h})^2\|_{L^\infty(\Omega)}$.

In the paper [39], a Gronwall-type lemma (Lemma 4.1) is specified which extracts a bound for the value $k(T)$ if an inequality like (4.95) is satisfied:

$$k_h(T)\leq Ce^{C_{i19}T}k_h(0)+C_{i17}\sqrt{2}h^kTe^{C_{i19}T}.$$

From this and the triangle inequality in conjunction with Lemma (4.5) the statement follows. □

4.3 Time Discretization

We divide the time interval $(0, T)$ into $N \in \mathbb{N}$ equally spaced subintervals by using the nodal points

$$0 =: t^0 < t^1 < t^2 < \ldots < t^N := T,$$

with $t^n = n\Delta t$, $n = 0, 1, 2, \ldots, N$ and $\Delta t := \frac{T}{N}$.

The Fully Discrete Scheme for the Discontinuous Galerkin Mehod

Given initial values $(E^0_{x,h}, E^0_{y,h}, H^0_{z,h}) \in U^k_h$ of the electric and magnetic field intensities. For all test functions $(\Phi_{1,h}, \Phi_{2,h}, \Phi_{3,h}) \in U^k_h$, the fully discrete electric and magnetic field intensities $(E^{n+1}_{x,h}, E^{n+1}_{y,h}, H^{n+\frac{1}{2}}_{z,h}) \in U^k_h$, $n = 1, 2, \ldots, N$ reads as

$$\begin{aligned}
&\int_{K_{i,j}} \frac{D^{n+1}_{x,h} - D^n_{x,h}}{\Delta t}\Phi_{1,h} - \int_{I_i} [(\hat{H}^{n+\frac{1}{2}}_{z,h}\Phi^-_{1,h})_{x,j+\frac{1}{2}} - (\hat{H}^{n+\frac{1}{2}}_{z,h}\Phi^+_{1,h})_{x,j-\frac{1}{2}}]dx \\
&+ \int_{K_{i,j}} H^{n+\frac{1}{2}}_{z,h}\partial_y\Phi_{1,h} - \int_{K_{i,j}} J^{n+\frac{1}{2}}_{x,h}\Phi_{1,h} = 0,
\end{aligned} \tag{4.96}$$

$$\begin{aligned}
&\int_{K_{i,j}} \frac{D^{n+1}_{y,h} - D^n_{y,h}}{\Delta t} + \int_{J_j} [(\hat{H}^{n+\frac{1}{2}}_{z,h}\Phi^-_{2,h})_{i+\frac{1}{2},y} - (\hat{H}^{n+\frac{1}{2}}_{z,h}\Phi^+_{2,h})_{i-\frac{1}{2},y}]dy \\
&- \int_{K_{i,j}} H^{n+\frac{1}{2}}_{z,h}\partial_x\Phi_{2,h} - \int_{K_{i,j}} J^{n+\frac{1}{2}}_{y,h}\Phi_{2,h} = 0,
\end{aligned} \tag{4.97}$$

$$\begin{aligned}
&\mu_0 \int_{K_{i,j}} \frac{H^{n+\frac{3}{2}}_{z,h} - H^{n+\frac{1}{2}}_{z,h}}{\Delta t}\Phi_{3,h} + \int_{J_j} [(\hat{E}^{n+1}_{y,h}\Phi^-_{3,h})_{i+\frac{1}{2},y} - (\hat{E}^{n+1}_{y,h}\Phi^+_{3,h})_{i-\frac{1}{2},y}]dy \\
&- \int_{K_{i,j}} E^{n+1}_{y,h}\partial_x\Phi_{3,h} - \int_{I_i} [(\hat{E}^{n+1}_{x,h}\Phi^-_{3,h})_{x,j+\frac{1}{2}} - (\hat{E}^{n+1}_{x,h}\Phi^+_{3,h})_{x,j-\frac{1}{2}}]dx \\
&+ \int_{K_{i,j}} E^{n+1}_{x,h}\partial_y\Phi_{3,h} = 0,
\end{aligned} \tag{4.98}$$

$$\begin{aligned}
&\int_{K_{i,j}} (D^{n+1}_{x,h} - D^n_{x,h})\Phi_{1,h} = \varepsilon_0(1 + \chi^{(1)}) \int_{K_{i,j}} (E^{n+1}_{x,h} - E^n_{x,h})\Phi_{1,h} \\
&+ \varepsilon_0\chi^{(3)} \int_{K_{i,j}} \Big[\frac{1}{2}\big[(E^{n+1}_{x,h})^2 + (E^n_{x,h})^2 + (E^{n+1}_{y,h})^2 + (E^n_{y,h})^2\big](E^{n+1}_{x,h} - E^n_{x,h})\Phi_{1,h} \\
&+ \big([(E^{n+1}_{x,h})^2 + (E^n_{x,h})^2](E^{n+1}_{x,h} - E^n_{x,h})\Phi_{1,h} \\
&+ [E^{n+1}_{x,h}E^{n+1}_{y,h} + E^n_{x,h}E^n_{y,h}](E^{n+1}_{y,h} - E^n_{y,h})\Phi_{1,h}\big)\Big],
\end{aligned} \tag{4.99}$$

$$\int_{K_{i,j}} (D_{y,h}^{n+1} - D_{y,h}^{n})\Phi_{2,h} = \varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} (E_{y,h}^{n+1} - E_{y,h}^{n})\Phi_{2,h}$$
$$+ \varepsilon_0 \chi^{(3)} \int_{K_{i,j}} \Big[\frac{1}{2}\big((E_{x,h}^{n+1})^2 + (E_{x,h}^{n})^2 + (E_{y,h}^{n+1})^2 + (E_{y,h}^{n})^2 \big)(E_{y,h}^{n+1} - E_{y,h}^{n})\Phi_{2,h}$$
$$+ \big([(E_{y,h}^{n+1})^2 + (E_{y,h}^{n})^2](E_{y,h}^{n+1} - E_{y,h}^{n})\Phi_{2,h}$$
$$+ [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n}E_{y,h}^{n}](E_{x,h}^{n+1} - E_{x,h}^{n})\Phi_{2,h} \big) \Big]. \tag{4.100}$$

The differences $D_{x,h}^{n+1} - D_{x,h}^{n}$ and $D_{y,h}^{n+1} - D_{y,h}^{n}$ play the role of auxiliary variables and the fluxes are defined by

$$\hat{E}_{x,h}^{n+1}(x, y_{j+\frac{1}{2}}) := E_{x,h}^{n+1,+}(x, y_{j+\frac{1}{2}}) \quad \forall j = 1,2,3,\dots,N_{y-1}, \tag{4.101}$$
$$\hat{E}_{x,h}^{n+1}(x, y_{\frac{1}{2}}) := \hat{E}_{x,h}^{n+1}(x, y_{N_y+\frac{1}{2}}) = 0, \tag{4.102}$$
$$\hat{E}_{y,h}^{n+1}(x_{i+\frac{1}{2}}, y) := E_{y,h}^{n+1,+}(x_{i+\frac{1}{2}}, y) \quad \forall i = 1,2,3,\dots,N_{x-1}, \tag{4.103}$$
$$\hat{E}_{y,h}^{n+1}(x_{\frac{1}{2}}, y) := \hat{E}_{y,h}^{n+1}(x_{N_x+\frac{1}{2}}, y) = 0, \tag{4.104}$$
$$\hat{H}_{z,h}^{n+\frac{1}{2}}(x, y_{j+\frac{1}{2}}) := H_{z,h}^{n+\frac{1}{2},-}(x, y_{j+\frac{1}{2}}) \quad \forall j = 1,2,3,\dots,N_y, \tag{4.105}$$
$$\hat{H}_{z,h}^{n+\frac{1}{2}}(x, y_{\frac{1}{2}}) := H_{z,h}^{n+\frac{1}{2},+}(x, y_{\frac{1}{2}}) + \frac{c_0}{2}\Big[E_{x,h}^{n+1}(x, y_{\frac{1}{2}}^{+}) + E_{x,h}^{n}(x, y_{\frac{1}{2}}^{+}) \Big], \tag{4.106}$$
$$\hat{H}_{z,h}^{n+\frac{1}{2}}(x_{i+\frac{1}{2}}, y) := H_{z,h}^{n+\frac{1}{2},-}(x_{i+\frac{1}{2}}, y) \quad \forall j = 1,2,3,\dots,N_x, \tag{4.107}$$
$$\hat{H}_{z,h}^{n+\frac{1}{2}}(x_{\frac{1}{2}}, y) := H_{z,h}^{n+\frac{1}{2},+}(x_{\frac{1}{2}}, y) - \frac{c_0}{2}\Big[E_{y,h}^{n+1}(x_{\frac{1}{2}}^{+}, y) + E_{x,h}^{n}(x_{\frac{1}{2}}^{+}, y) \Big]. \tag{4.108}$$

Note that the PEC conditions (4.11) are enforced, $E_{x,h}^{n+1}(x, y_{\frac{1}{2}}^{+}) = E_{x,h}^{n+1}(x, y_{\frac{1}{2}}^{+}) - E_{x,h}^{n+1}(x, y_{\frac{1}{2}}^{-}) = [\![E_{x,h}^{n+1}(x, y_{\frac{1}{2}})]\!]$ in the equation (4.106) and the same for the other artificial viscosity in the equation (4.108). The boundary terms are defined

$$\sigma_{Ih} := - \int_{I_i} [(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1} + E_{x,h}^{n})^{-})_{x,j+\frac{1}{2}} - (\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1} + E_{x,h}^{n})^{+})_{x,j-\frac{1}{2}}]dx$$
$$- \int_{I_i} [(\hat{E}_{x,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^{-})_{x,j+\frac{1}{2}} - (\hat{E}_{x,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^{+})_{x,j-\frac{1}{2}}]dx, \tag{4.109}$$

and

$$\sigma_{Jh} := \int_{J_j} [(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1} + E_{y,h}^{n})^{-})_{i+\frac{1}{2},y} - (\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1} + E_{y,h}^{n})^{+})_{i-\frac{1}{2},y}]dy$$
$$+ \int_{J_j} [(\hat{E}_{y,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^{-})_{i+\frac{1}{2},y} - (\hat{E}_{y,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^{+})_{i-\frac{1}{2},y}]dy. \tag{4.110}$$

Lemma 4.11 *For $n = 1, 2, \ldots, N$, with the fluxes (4.101)-(4.108), we have*

$$
\begin{aligned}
&\sum_{n=0}^{N} \sum_{1\le i\le N_x} \sum_{1\le j\le N_y} \int_{K_{i,j}} \Big[H_{z,h}^{n+\frac{1}{2}} \partial_y (E_{x,h}^{n+1} + E_{x,h}^{n}) + E_{x,h}^{n+1} \partial_y (H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}}) \Big] \\
&+ \sum_{n=0}^{N} \sum_{1\le i\le N_x} \sum_{1\le j\le N_y} \sigma_{Ih} \\
&= \sum_{1\le i\le N_{y-1}} \Big[\int_r^s \Big(E_{x,h}^{n+1,+} [\![H_{z,h}^{n+\frac{3}{2}}]\!] \Big)_{x,j+\frac{1}{2}} - \int_r^s \Big(E_{x,h}^{0,+} [\![H_{z,h}^{\frac{1}{2}}]\!] \Big)_{x,j+\frac{1}{2}} \Big] \\
&+ \sum_{1\le i\le N_x} \sum_{1\le j\le N_y} \int_{K_{i,j}} \Big[E_{x,h}^{n+1} \partial_y H_{z,h}^{n+\frac{3}{2}} - E_{x,h}^{0} \partial_y H_{z,h}^{\frac{1}{2}} \Big] \\
&+ \frac{c_0}{2} \sum_{n=0}^{N} \int_r^s (E_{x,h}^{n+1,+} + E_{x,h}^{n,+})^2_{x,\frac{1}{2}}. \qquad (4.111)
\end{aligned}
$$

Proof: For details see [93, Lemma 4.1]. $\square$

Lemma 4.12 *For $n = 1, 2, \ldots, N$, with the fluxes (4.101)-(4.108), we have*

$$
\begin{aligned}
&-\sum_{n=0}^{N} \sum_{1\le i\le N_x} \sum_{1\le j\le N_y} \int_{K_{i,j}} \Big[H_{z,h}^{n+\frac{1}{2}} \partial_x (E_{y,h}^{n+1} + E_{y,h}^{n}) + E_{y,h}^{n+1} \partial_x (H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}}) \Big] \\
&+ \sum_{n=0}^{N} \sum_{1\le i\le N_x} \sum_{1\le j\le N_y} \sigma_{Jh} \\
&= \sum_{1\le i\le N_{x-1}} \Big[-\int_p^q \Big(E_{y,h}^{n+1,+} [\![H_{z,h}^{n+\frac{3}{2}}]\!] \Big)_{i+\frac{1}{2},y} + \int_p^q \Big(E_{y,h}^{0,+} [\![H_{z,h}^{\frac{1}{2}}]\!] \Big)_{i+\frac{1}{2},y} \Big] \\
&+ \sum_{1\le i\le N_x} \sum_{1\le j\le N_y} \int_{K_{i,j}} \Big[-E_{y,h}^{n+1} \partial_x H_{z,h}^{n+\frac{3}{2}} + E_{y,h}^{0} \partial_x H_{z,h}^{\frac{1}{2}} \Big] \\
&+ \frac{c_0}{2} \sum_{n=0}^{N} \int_p^q (E_{y,h}^{n+1,+} + E_{y,h}^{n,+})^2_{\frac{1}{2},y} \qquad (4.112)
\end{aligned}
$$

Proof: For details see [93, Lemma 4.2]. $\square$

4.3.1 The Nonlinear Electromagnetic Energy of Discontinuous Galerkin Method at the Fully Discrete Level

The nonlinear electromagnetic energy for the fully discrete approximation (i.e. both in space and time) of the system (4.96)–(4.100) at t^n, $n = 0, 1, 2, \ldots, N$, is defined by

$$\begin{aligned}\mathcal{E}_h^n :=& \|E_{x,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{n+\frac{3}{2}}\|^2_{\mu_0} \\ &+ \frac{3}{2}\|(E_{x,h}^{n+1})^2 + (E_{y,h}^{n+1})^2\|^2_{\varepsilon_0\chi^{(3)}}.\end{aligned}$$

In analogy to the conservativity and boundedness results for the continuous and semi-discrete nonlinear electromagnetic energy (Thms. 4.1, 4.3), in this section we will show that the fully discrete nonlinear electromagnetic energy of the system (4.96)–(4.100) at the final time step N is conserved and bounded, too.

Theorem 4.13 *Let $(E_{x,h}^{n+1}, E_{y,h}^{n+1}, H_{z,h}^{n+\frac{1}{2}})$ be the fully discrete solution of (4.96)–(4.100). Then, for sufficiently small Δt, h and $\Delta t/h^2$, there exists a constant $C > 0$ independent of Δt and h for vanishing current density such that*

$$\begin{aligned}\mathcal{E}_h^N =& \|E_{x,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} \\ &+ \frac{3}{2}\Big[\|(E_{x,h}^{N+1})^2 + (E_{y,h}^{N+1})^2\|^2_{\varepsilon_0\chi^{(3)}}\Big] \\ =& \|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0} \\ &+ \frac{3}{2}\Big[\|(E_{x,h}^{0})^2 + (E_{y,h}^{0})^2\|^2_{\varepsilon_0\chi^{(3)}}\Big],\end{aligned}$$

and for non-zero current density

$$\begin{aligned}\mathcal{E}_h^N =& \|E_{x,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} \\ &+ \frac{3}{2}\Big[\|(E_{x,h}^{N+1})^2 + (E_{y,h}^{N+1})^2\|^2_{\varepsilon_0\chi^{(3)}}\Big] \leq C.\end{aligned}$$

Proof: Taking $\Phi_{1,h} = (E_{x,h}^{n+1} + E_{x,h}^{n})$ in the equation (4.99), we have

$$\begin{aligned}
&\int_{K_{i,j}} (D_{x,h}^{n+1} - D_{x,h}^{n})(E_{x,h}^{n+1} + E_{x,h}^{n}) \\
&= \varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} (E_{x,h}^{n+1} - E_{x,h}^{n})(E_{x,h}^{n+1} + E_{x,h}^{n}) \\
&\quad + \varepsilon_0 \chi^{(3)} \int_{K_{i,j}} \Big[\frac{1}{2}\big[(E_{x,h}^{n+1})^2 + (E_{x,h}^{n})^2 + (E_{y,h}^{n+1})^2 + (E_{y,h}^{n})^2\big] \\
&\quad \times (E_{x,h}^{n+1} - E_{x,h}^{n})(E_{x,h}^{n+1} + E_{x,h}^{n}) \\
&\quad + \Big([(E_{x,h}^{n+1})^2 + (E_{x,h}^{n})^2](E_{x,h}^{n+1} - E_{x,h}^{n})(E_{x,h}^{n+1} + E_{x,h}^{n}) \\
&\quad + [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n}E_{y,h}^{n}](E_{y,h}^{n+1} - E_{y,h}^{n})(E_{x,h}^{n+1} + E_{x,h}^{n})\Big)\Big]
\end{aligned} \tag{4.113}$$

Taking $\Phi_{2,h} = (E_{y,h}^{n+1} + E_{y,h}^{n})$ in the equation (4.100), we have

$$\begin{aligned}
&\int_{K_{i,j}} (D_{y,h}^{n+1} - D_{y,h}^{n})(E_{y,h}^{n+1} + E_{y,h}^{n}) \\
&= \varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} (E_{y,h}^{n+1} - E_{y,h}^{n})(E_{y,h}^{n+1} + E_{y,h}^{n}) \\
&\quad + \varepsilon_0 \chi^{(3)} \int_{K_{i,j}} \Big[\frac{1}{2}\big[(E_{x,h}^{n+1})^2 + (E_{x,h}^{n})^2 + (E_{y,h}^{n+1})^2 + (E_{y,h}^{n})^2\big] \\
&\quad \times (E_{y,h}^{n+1} - E_{y,h}^{n})(E_{y,h}^{n+1} + E_{y,h}^{n}) \\
&\quad + \Big([(E_{y,h}^{n+1})^2 + (E_{y,h}^{n})^2](E_{y,h}^{n+1} - E_{y,h}^{n})(E_{y,h}^{n+1} + E_{y,h}^{n}) \\
&\quad + [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n}E_{y,h}^{n}](E_{x,h}^{n+1} - E_{x,h}^{n})(E_{y,h}^{n+1} + E_{y,h}^{n})\Big)\Big].
\end{aligned} \tag{4.114}$$

Adding the equations (4.113) and (4.114), we see that

$$\begin{aligned}
&\int_{K_{i,j}} (D_{x,h}^{n+1} - D_{x,h}^{n})(E_{x,h}^{n+1} + E_{x,h}^{n}) + \int_{K_{i,j}} (D_{y,h}^{n+1} - D_{y,h}^{n})(E_{y,h}^{n+1} + E_{y,h}^{n}) \\
&= \varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} [(E_{x,h}^{n+1})^2 - (E_{x,h}^{n})^2 + (E_{y,h}^{n+1})^2 - (E_{y,h}^{n})^2] \\
&\quad + \varepsilon_0\chi^{(3)} \int_{K_{i,j}} \Big[\frac{1}{2}\big[(E_{x,h}^{n+1})^2 + (E_{x,h}^{n})^2 + (E_{y,h}^{n+1})^2 + (E_{y,h}^{n})^2\big] \\
&\quad \times [(E_{x,h}^{n+1})^2 - (E_{x,h}^{n})^2 + (E_{y,h}^{n+1})^2 - (E_{y,h}^{n})^2] \\
&\quad + [(E_{x,h}^{n+1})^2 + (E_{x,h}^{n})^2][(E_{x,h}^{n+1})^2 - (E_{x,h}^{n})^2] + [(E_{y,h}^{n+1})^2 + (E_{y,h}^{n})^2] \\
&\quad \times [(E_{y,h}^{n+1})^2 - (E_{y,h}^{n})^2] \\
&\quad + [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n}E_{y,h}^{n}][E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n}E_{y,h}^{n+1} - E_{x,h}^{n+1}E_{y,h}^{n} - E_{y,h}^{n}E_{x,h}^{n}] \\
&\quad + [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n}E_{y,h}^{n}][E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n+1}E_{y,h}^{n} - E_{x,h}^{n}E_{y,h}^{n+1} - E_{x,h}^{n}E_{y,h}^{n}]\Big].
\end{aligned}$$

Furthermore we can rewrite

$$\begin{aligned}
&\int_{K_{i,j}} (D_{x,h}^{n+1} - D_{x,h}^{n})(E_{x,h}^{n+1} + E_{x,h}^{n}) + \int_{K_{i,j}} (D_{y,h}^{n+1} - D_{y,h}^{n})(E_{y,h}^{n+1} + E_{y,h}^{n}) \\
&= \varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} [(E_{x,h}^{n+1})^2 - (E_{x,h}^{n})^2 + (E_{y,h}^{n+1})^2 - (E_{y,h}^{n})^2] \\
&\quad + \varepsilon_0\chi^{(3)} \int_{K_{i,j}} \Big[\frac{1}{2}\big[[(E_{x,h}^{n+1})^2 + (E_{x,h}^{n})^2][(E_{x,h}^{n+1})^2 - (E_{x,h}^{n})^2] \\
&\quad + [(E_{x,h}^{n+1})^2 + (E_{x,h}^{n})^2][(E_{y,h}^{n+1})^2 - (E_{y,h}^{n})^2] \\
&\quad + [(E_{y,h}^{n+1})^2 + (E_{y,h}^{n})^2][(E_{x,h}^{n+1})^2 - (E_{x,h}^{n})^2] \\
&\quad + [(E_{y,h}^{n+1})^2 + (E_{y,h}^{n})^2][(E_{y,h}^{n+1})^2 - (E_{y,h}^{n})^2]\big] \\
&\quad + [(E_{x,h}^{n+1})^4 - (E_{x,h}^{n})^4] + [(E_{y,h}^{n+1})^4 - (E_{y,h}^{n})^4] \\
&\quad + [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n}E_{y,h}^{n}][E_{x,h}^{n+1}E_{y,h}^{n+1} - E_{y,h}^{n}E_{x,h}^{n}] \\
&\quad + [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n}E_{y,h}^{n}][E_{x,h}^{n+1}E_{y,h}^{n+1} - E_{x,h}^{n}E_{y,h}^{n}]\Big].
\end{aligned}$$

This implies that

$$
\begin{aligned}
&\int_{K_{i,j}} (D_{x,h}^{n+1} - D_{x,h}^{n})(E_{x,h}^{n+1} + E_{x,h}^{n}) + \int_{K_{i,j}} (D_{y,h}^{n+1} - D_{y,h}^{n})(E_{y,h}^{n+1} + E_{y,h}^{n}) \\
&= \varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} [(E_{x,h}^{n+1})^2 - (E_{x,h}^{n})^2 + (E_{y,h}^{n+1})^2 - (E_{y,h}^{n})^2] \\
&\quad + \varepsilon_0\chi^{(3)} \int_{K_{i,j}} \Big[\frac{1}{2}\big[[(E_{x,h}^{n+1})^4 - (E_{x,h}^{n})^4] + [(E_{x,h}^{n+1})^2(E_{y,h}^{n+1})^2 - (E_{x,h}^{n+1})^2(E_{y,h}^{n})^2 \\
&\quad + (E_{x,h}^{n})^2(E_{y,h}^{n+1})^2 - (E_{x,h}^{n})^2(E_{y,h}^{n})^2] + [(E_{y,h}^{n+1})^2(E_{x,h}^{n+1})^2 - (E_{y,h}^{n+1})^2(E_{x,h}^{n})^2 \\
&\quad + (E_{y,h}^{n})^2 E_{x,h}^{n+1})^2 - (E_{y,h}^{n})^2(E_{x,h}^{n})^2] + [(E_{y,h}^{n+1})^4 - (E_{y,h}^{n})^4]\big] \\
&\quad + [(E_{x,h}^{n+1})^4 - (E_{x,h}^{n})^4] + [(E_{y,h}^{n+1})^4 - (E_{y,h}^{n})^4] \\
&\quad + 2[(E_{x,h}^{n+1}E_{y,h}^{n+1})^2 - (E_{y,h}^{n}E_{x,h}^{n})^2]\Big].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_{K_{i,j}} (D_{x,h}^{n+1} - D_{x,h}^{n})(E_{x,h}^{n+1} + E_{x,h}^{n}) + \int_{K_{i,j}} (D_{y,h}^{n+1} - D_{y,h}^{n})(E_{y,h}^{n+1} + E_{y,h}^{n}) \\
&= \varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} [(E_{x,h}^{n+1})^2 - (E_{x,h}^{n})^2 + (E_{y,h}^{n+1})^2 - (E_{y,h}^{n})^2] \\
&\quad + \varepsilon_0\chi^{(3)} \int_{K_{i,j}} \Big[\frac{1}{2}\big[[(E_{x,h}^{n+1})^4 - (E_{x,h}^{n})^4] + [(E_{y,h}^{n+1})^4 - (E_{y,h}^{n})^4] \\
&\quad + 2[(E_{y,h}^{n+1})^2(E_{x,h}^{n+1})^2 - (E_{y,h}^{n})^2(E_{x,h}^{n})^2]\big] \\
&\quad + [(E_{x,h}^{n+1})^4 - (E_{x,h}^{n})^4] + [(E_{y,h}^{n+1})^4 - (E_{y,h}^{n})^4] + 2[(E_{x,h}^{n+1}E_{y,h}^{n+1})^2 - (E_{y,h}^{n}E_{x,h}^{n})^2]\Big].
\end{aligned}
$$

Finally we can rewrite

$$
\begin{aligned}
&\int_{K_{i,j}} (D_{x,h}^{n+1} - D_{x,h}^{n})(E_{x,h}^{n+1} + E_{x,h}^{n}) + \int_{K_{i,j}} (D_{y,h}^{n+1} - D_{y,h}^{n})(E_{y,h}^{n+1} + E_{y,h}^{n}) \\
&= \varepsilon_0(1+\chi^{(1)}) \int_{K_{i,j}} [(E_{x,h}^{n+1})^2 - (E_{x,h}^{n})^2 + (E_{y,h}^{n+1})^2 - (E_{y,h}^{n})^2] \\
&+ \varepsilon_0\chi^{(3)}\frac{3}{2} \int_{K_{i,j}} \big[[(E_{x,h}^{n+1})^4 - (E_{x,h}^{n})^4] + [(E_{y,h}^{n+1})^4 - (E_{y,h}^{n})^4] \\
&+ 3\varepsilon_0\chi^{(3)} \int_{K_{i,j}} [(E_{x,h}^{n+1}E_{y,h}^{n+1})^2 - (E_{y,h}^{n}E_{x,h}^{n})^2].
\end{aligned}
\tag{4.115}
$$

Taking $\Phi_{1,h} = 2\Delta t(E^{n+1}_{x,h} + E^n_{x,h})$ in the equation (4.96), we have

$$
\begin{aligned}
&2\int_{K_{i,j}} (D^{n+1}_{x,h} - D^n_{x,h})(E^{n+1}_{x,h} + E^n_{x,h}) - 2\Delta t \int_{I_i} [(\hat{H}^{n+\frac{1}{2}}_{z,h}(E^{n+1}_{x,h} + E^n_{x,h})^-)_{x,j+\frac{1}{2}} \\
&- (\hat{H}^{n+\frac{1}{2}}_{z,h}(E^{n+1}_{x,h} + E^n_{x,h})^+)_{x,j-\frac{1}{2}}]dx + 2\Delta t \int_{K_{i,j}} H^{n+\frac{1}{2}}_{z,h} \partial_y (E^{n+1}_{x,h} + E^n_{x,h}) \\
&- 2\Delta t \int_{K_{i,j}} J^{n+\frac{1}{2}}_{x,h}(E^{n+1}_{x,h} + E^n_{x,h}) = 0,
\end{aligned} \tag{4.116}
$$

Taking $\Phi_{2,h} = 2\Delta t(E^{n+1}_{y,h} + E^n_{y,h})$ in the equation (4.97), we have

$$
\begin{aligned}
&2\int_{K_{i,j}} (D^{n+1}_{y,h} - D_{y,h)})(E^{n+1}_{y,h} + E^n_{y,h}) + 2\Delta t \int_{J_j} [(\hat{H}^{n+\frac{1}{2}}_{z,h}(E^{n+1}_{y,h} + E^n_{y,h})^-)_{i+\frac{1}{2},y} \\
&- (\hat{H}^{n+\frac{1}{2}}_{z,h}(E^{n+1}_{y,h} + E^n_{y,h})^+)_{i-\frac{1}{2},y}]dy - 2\Delta t \int_{K_{i,j}} H^{n+\frac{1}{2}}_{z,h} \partial_x (E^{n+1}_{y,h} + E^n_{y,h}) \\
&- 2\Delta t \int_{K_{i,j}} J^{n+\frac{1}{2}}_{y,h}(E^{n+1}_{y,h} + E^n_{y,h}) = 0,
\end{aligned} \tag{4.117}
$$

Taking $\Phi_{3,h} = 2\Delta t(H^{n+\frac{3}{2}}_{z,h} + H^{n+\frac{1}{2}}_{z,h})$ in the equation (4.98), we have

$$
\begin{aligned}
&2\mu_0 \int_{K_{i,j}} (H^{n+\frac{3}{2}}_{z,h} - H^{n+\frac{1}{2}}_{z,h})(H^{n+\frac{3}{2}}_{z,h} + H^{n+\frac{1}{2}}_{z,h}) + 2\Delta t \int_{J_j} [(\hat{E}^{n+1}_{y,h}(H^{n+\frac{3}{2}}_{z,h} + H^{n+\frac{1}{2}}_{z,h})^-)_{i+\frac{1}{2},y} \\
&- (\hat{E}^{n+1}_{y,h}(H^{n+\frac{3}{2}}_{z,h} + H^{n+\frac{1}{2}}_{z,h})^+)_{i-\frac{1}{2},y}]dy - 2\Delta t \int_{K_{i,j}} E^{n+1}_{y,h} \partial_x (H^{n+\frac{3}{2}}_{z,h} + H^{n+\frac{1}{2}}_{z,h}) \\
&- 2\Delta t \int_{I_i} [(\hat{E}^{n+1}_{x,h}(H^{n+\frac{3}{2}}_{z,h} + H^{n+\frac{1}{2}}_{z,h})^-)_{x,j+\frac{1}{2}} - (\hat{E}^{n+1}_{x,h}(H^{n+\frac{3}{2}}_{z,h} + H^{n+\frac{1}{2}}_{z,h})^+)_{x,j-\frac{1}{2}}]dx \\
&+ 2\Delta t \int_{K_{i,j}} E^{n+1}_{x,h} \partial_y (H^{n+\frac{3}{2}}_{z,h} + H^{n+\frac{1}{2}}_{z,h}) = 0.
\end{aligned} \tag{4.118}
$$

Adding the equations (4.117)–(4.118), substituting the equation (4.115), we obtain

$$\begin{aligned}
&2\varepsilon_0(1+\chi^{(1)})\int_{K_{i,j}}[(E_{x,h}^{n+1})^2-(E_{x,h}^{n})^2+(E_{y,h}^{n+1})^2-(E_{y,h}^{n})^2]\\
&+2\mu_0\int_{K_{i,j}}[(H_{z,h}^{n+\frac{3}{2}})^2-(H_{z,h}^{n+\frac{1}{2}})^2]\\
&+\varepsilon_0\chi^{(3)}3\int_{K_{i,j}}\Big[[(E_{x,h}^{n+1})^4-(E_{x,h}^{n})^4]+[(E_{y,h}^{n+1})^4-(E_{y,h}^{n})^4]\Big]\\
&+6\varepsilon_0\chi^{(3)}\int_{K_{i,j}}[(E_{x,h}^{n+1}E_{y,h}^{n+1})^2-(E_{y,h}^{n}E_{x,h}^{n})^2]\\
&-2\Delta t\int_{I_i}[(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1}+E_{x,h}^{n})^-)_{x,j+\frac{1}{2}}-(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1}+E_{x,h}^{n})^+)_{x,j-\frac{1}{2}}]dx\\
&+2\Delta t\int_{K_{i,j}}H_{z,h}^{n+\frac{1}{2}}\partial_y(E_{x,h}^{n+1}+E_{x,h}^{n})\\
&+2\Delta t\int_{J_j}[(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1}+E_{y,h}^{n})^-)_{i+\frac{1}{2},y}-(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1}+E_{y,h}^{n})^+)_{i-\frac{1}{2},y}]dy\\
&-2\Delta t\int_{K_{i,j}}H_{z,h}^{n+\frac{1}{2}}\partial_x(E_{y,h}^{n+1}+E_{y,h}^{n})+2\Delta t\int_{J_j}[(\hat{E}_{y,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})^-)_{i+\frac{1}{2},y}\\
&-(\hat{E}_{y,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})^+)_{i-\frac{1}{2},y}]dy-2\Delta t\int_{K_{i,j}}E_{y,h}^{n+1}\partial_x(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})\\
&-2\Delta t\int_{I_i}[(\hat{E}_{x,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})^-)_{x,j+\frac{1}{2}}-(\hat{E}_{x,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})^+)_{x,j-\frac{1}{2}}]dx\\
&+2\Delta t\int_{K_{i,j}}E_{x,h}^{n+1}\partial_y(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})\\
&=2\Delta t\int_{K_{i,j}}J_{y,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1}+E_{y,h}^{n})+2\Delta t\int_{K_{i,j}}J_{x,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1}+E_{x,h}^{n}). \qquad (4.119)
\end{aligned}$$

Summing up over the $1 \leq i \leq N_x$, $1 \leq i \leq N_y$, and with respect to time from $n = 1$ to N, and using the Lemmas 4.11–4.12 we arrive at

$$
\begin{aligned}
&2\Big[\|E_{x,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} - \|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} \\
&+ \|H_{z,h}^{n+\frac{3}{2}}\|^2_{\mu_0} - \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big] + 3\Big[\|E_{x,h}^{N+1}\|^4_{\varepsilon_0\chi^{(3)}} - \|E_{x,h}^{0}\|^4_{\varepsilon_0\chi^{(3)}} \\
&+ \|E_{y,h}^{N+1}\|^4_{\varepsilon_0\chi^{(3)}} - \|E_{y,h}^{0}\|^4_{\varepsilon_0\chi^{(3)}}]\Big] + 6[\|E_{x,h}^{n+1}E_{y,h}^{n+1}\|^2_{\varepsilon_0\chi^{(3)}} - \|E_{y,h}^{0}E_{x,h}^{0}\|^2_{\varepsilon_0\chi^{(3)}}] \\
&= 2\Delta t\sum_{n=0}^{N}\int_\Omega J_{y,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1} + E_{y,h}^{n}) + 2\Delta t\sum_{n=0}^{N}\int_\Omega J_{x,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1} + E_{x,h}^{n}) \\
&- 2B_y\Big(E_{x,h}^{N+1}, H_{z,h}^{N+\frac{3}{2}}\Big) + 2B_y\Big(E_{x,h}^{0}, H_{z,h}^{\frac{1}{2}}\Big) \\
&+ 2B_x\Big(E_{y,h}^{N+1}, H_{z,h}^{N+\frac{3}{2}}\Big) - 2B_x\Big(E_{y,h}^{0}, H_{z,h}^{\frac{1}{2}}\Big).
\end{aligned}
\tag{4.120}
$$

The bilinear forms are defined as

$$
\begin{aligned}
B_x\Big(E_{y,h}^{n+1}, H_{z,h}^{n+\frac{3}{2}}\Big) &= \Delta t\Big[\sum_{1\leq i\leq N_x}\sum_{1\leq i\leq N_y}\int_{K_{i,j}} E_{y,h}^{n+1}\partial_x H_{z,h}^{n+\frac{3}{2}} \\
&+ \sum_{1\leq i\leq N_x-1}\int_p^q (E_{y,h})^+_{i+\frac{1}{2}}[\![H_{z,h}^{n+\frac{3}{2}}]\!]_{i+\frac{1}{2}}dy\Big], \qquad (4.121)\\
B_y\Big(E_{x,h}^{n+1}, H_{z,h}^{n+\frac{3}{2}}\Big) &= \Delta t\Big[\sum_{1\leq i\leq N_x}\sum_{1\leq i\leq N_y}\int_{K_{i,j}} E_{x,h}^{n+1}\partial_y H_{z,h}^{n+\frac{3}{2}} \\
&+ \sum_{1\leq i\leq N_y-1}\int_r^s (E_{x,h})^+_{j+\frac{1}{2}}[\![H_{z,h}^{n+\frac{3}{2}}]\!]_{j+\frac{1}{2}}dx\Big] \qquad (4.122)
\end{aligned}
$$

(cf. [137, equation (4.1)]). Using an inverse estimate (cf. [137, Lemma 4.1]), we have that

$$
B_y\Big(E_{x,h}^{n+1}, H_{z,h}^{n+\frac{3}{2}}\Big) \leq 2\Delta t C_{INV}\frac{C_{\varepsilon\mu}}{h}\sqrt{\varepsilon_0(1+\chi^{(1)})}\|E_{x,h}^{n+1}\|\,\sqrt{\mu_0}\|H_{z,h}^{n+\frac{3}{2}}\|,
$$

where C_{INV} is a positive constant that is independent of h and Δt, and $C_{\varepsilon\mu} = \frac{1}{\sqrt{\varepsilon_0\mu_0(1+\chi^{(1)})}}$. The right-hand side is estimated by means of the inequality (2) from Lemma 5.1, where the parameter $\alpha > 0$ will be determined later:

$$
\begin{aligned}
B_y\Big(E_{x,h}^{n+1}, H_{z,h}^{n+\frac{3}{2}}\Big) &\leq 2\Delta t C_{INV}\frac{C_{\varepsilon\mu}}{h}\|E_{x,h}^{n+1}\|_{\varepsilon_0(1+\chi^{(1)})}\,\|H_{z,h}^{n+\frac{3}{2}}\|_{\mu_0} \\
&\leq \Big[\alpha\|E_{x,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \Big(\Delta t C_{INV}\frac{C_{\varepsilon\mu}}{\alpha h}\Big)^2\|H_{z,h}^{n+\frac{3}{2}}\|^2_{\mu_0}\Big].
\end{aligned}
\tag{4.123}
$$

Similarly we get (with the same parameter α)

$$B_x\Big(E_{x,y}^{n+1}, H_{z,h}^{n+\frac{3}{2}}\Big) \le \Big[\alpha \|E_{y,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \Big(\Delta t C_{INV}\frac{C_{\varepsilon\mu}}{\alpha h}\Big)^2 \|H_{z,h}^{n+\frac{3}{2}}\|^2_{\mu_0}\Big], \tag{4.124}$$

and

$$B_y\Big(E_{x,h}^{0}, H_{z,h}^{\frac{1}{2}}\Big) \le \Delta t C_{INV}\frac{C_{\varepsilon\mu}}{h}\Big[\|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big], \tag{4.125}$$

$$B_x\Big(E_{y,h}^{0}, H_{z,h}^{\frac{1}{2}}\Big) \le \Delta t C_{INV}\frac{C_{\varepsilon\mu}}{h}\Big[\|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big]. \tag{4.126}$$

The first two terms from the right-hand side of equation (4.120) are estimated by means of the inequality (2) from Lemma 5.1. This gives

$$\begin{aligned}
&2\Delta t \sum_{n=0}^{N} \int_\Omega J_{x,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1} + E_{x,h}^{n}) \\
&= \Delta t \sum_{n=0}^{N} \int_\Omega [\varepsilon_0(1+\chi^{(1)})]^{-1/2} J_{x,h}^{n+\frac{1}{2}}([\varepsilon_0(1+\chi^{(1)})]^{1/2}(E_{x,h}^{n+1} + E_{x,h}^{n}) \\
&\le \Delta t \sum_{n=0}^{N} \|J_{x,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \Delta t \sum_{n=0}^{N} \varepsilon_0(1+\chi^{(1)})\|E_{x,h}^{n+1} + E_{x,h}^{n}\|^2 \\
&\le \Delta t \sum_{n=0}^{N} \|J_{x,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + 2\Delta t \sum_{n=0}^{N} \varepsilon_0(1+\chi^{(1)})[\|E_{x,h}^{n+1}\|^2 + \|E_{x,h}^{n}\|^2] \\
&\le \Delta t \sum_{n=0}^{N} \|J_{x,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + 4\Delta t \sum_{n=0}^{N} \varepsilon_0(1+\chi^{(1)})\|E_{x,h}^{n+1}\|^2,
\end{aligned} \tag{4.127}$$

and

$$\begin{aligned}
&2\Delta t \sum_{n=0}^{N} \int_\Omega J_{y,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1} + E_{y,h}^{n}) \\
&\le \Delta t \sum_{n=0}^{N} \|J_{y,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + 4\Delta t \sum_{n=0}^{N} \varepsilon_0(1+\chi^{(1)})\|E_{y,h}^{n+1}\|^2.
\end{aligned} \tag{4.128}$$

Finally, using the estimates (4.127), (4.128), (4.123), (4.124), (4.125) and (4.126) in (4.120), we obtain

$$\begin{aligned}
&2\Big[\|E_{x,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0}\Big] \\
&+ 3\Big[\|(E_{x,h}^{N+1})^2\|^2_{\varepsilon_0\chi^{(3)}} + \|(E_{y,h}^{N+1})^2\|^2_{\varepsilon_0\chi^{(3)}}\Big] + 6\|E_{x,h}^{N+1}E_{y,h}^{N+1}\|^2_{\varepsilon_0\chi^{(3)}} \\
&\le 4\Delta t \sum_{n=0}^{N} \Big[\|E_{x,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big] \\
&+ 2\alpha\Big[\|E_{x,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big] + 4\Big(\Delta t C_{INV}\frac{C_{\varepsilon\mu}}{\alpha h}\Big)^2\|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} \\
&+ \Delta t \sum_{n=0}^{N} \|\Big[\|J_{x,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \|J_{y,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}}\Big] \\
&+ \Delta t C_{INV}\frac{C_{\varepsilon\mu}}{h}\Big[\|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big] \\
&+ 2\Big[\|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big] \\
&+ 3\big[\|E_{x,h}^{0}\|^4_{\varepsilon_0\chi^{(3)}}\big] + \|E_{y,h}^{0}\|^4_{\varepsilon_0\chi^{(3)}}\big] + 6\|E_{y,h}^{0}E_{x,h}^{0}\|^2_{\varepsilon_0\chi^{(3)}},
\end{aligned} \tag{4.129}$$

Furthermore we can rewrite this inequality as

$$\begin{aligned}
&\|E_{x,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} \\
&+ \frac{3}{2}\|(E_{x,h}^{N+1})^2 + (E_{y,h}^{N+1})^2\|^2_{\varepsilon_0\chi^{(3)}} \\
&\le 2\Delta t \sum_{n=0}^{N} \Big[\|E_{x,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big] \\
&+ \alpha\Big[\|E_{x,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big] + 2\Big(\Delta t C_{INV}\frac{C_{\varepsilon\mu}}{\alpha h}\Big)^2\|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} \\
&+ \Delta t\frac{1}{2} \sum_{n=0}^{N} \|\Big[\|J_{x,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \|J_{y,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}}\Big] \\
&+ \Delta t C_{INV}\frac{C_{\varepsilon\mu}}{2h}\Big[\|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big] \\
&+ \|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0} + \frac{3}{2}\|(E_{x,h}^{0})^2 + (E_{y,h}^{0})^2\|^2_{\varepsilon_0\chi^{(3)}}.
\end{aligned}$$

Now we chose $\alpha = 1/2$ and move the corresponding terms to the left-hand side. If the condition

$$\frac{\Delta t}{h} \le \frac{1}{4C_{INV}C_{\varepsilon\mu}}$$

is satisfied, we obtain

$$
\begin{aligned}
&\|E_{x,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} \\
&+ 3\|(E_{x,h}^{N+1})^2 + (E_{y,h}^{N+1})^2\|^2_{\varepsilon_0\chi^{(3)}} \\
&\leq 4\Delta t \sum_{n=0}^{N} \Big[\|E_{x,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})}\Big] \\
&+ \Delta t \sum_{n=0}^{N} \| \Big[\|J_{x,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \|J_{y,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}}\Big] \\
&+ \frac{5}{4}\Big[\|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big] \\
&+ 3\|(E_{x,h}^{0})^2 + (E_{y,h}^{0})^2\|^2_{\varepsilon_0\chi^{(3)}}.
\end{aligned} \tag{4.130}
$$

Now we employ the Gronwall's inequality (Lemma 5.3) with

$$
\begin{aligned}
\delta &:= \Delta t \geq 0, \\
g_0 &:= \frac{5}{4}\Big[\|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big] \\
&\quad + 3\|(E_{x,h}^{0})^2 + (E_{y,h}^{0})^2\|^2_{\varepsilon_0\chi^{(3)}} \geq 0, \\
a_n &:= \|E_{x,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{n+\frac{3}{2}}\|^2_{\mu_0} \\
&\quad + 3\|(E_{x,h}^{n+1})^2 + (E_{y,h}^{n+1})^2\|^2_{\varepsilon_0\chi^{(3)}} \geq 0 \quad \text{for } n \in \mathbb{N}_0, \\
b_n &:= 0 \quad \text{for } n \in \mathbb{N}_0, \\
c_0 &:= 0, \\
c_n &:= \|J_{x,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \|J_{y,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} \geq 0 \quad \text{for } n \in \mathbb{N}, \\
\gamma_n &:= 4 \geq 0 \quad \text{for } n \in \mathbb{N}_0.
\end{aligned}
$$

Then the condition $\gamma_n\,\delta < 1$ corresponds to $\Delta t < \frac{1}{4}$, and we obtain

$$
\begin{aligned}
&\|E_{x,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} \\
&+ 3\|(E_{x,h}^{N+1})^2 + (E_{y,h}^{N+1})^2\|^2_{\varepsilon_0\chi^{(3)}} \\
&\leq \Bigg(\Delta t \sum_{n=1}^{N} \Big[J_{x,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \|J_{y,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}}\Big] \\
&\quad + \frac{5}{4}\Big[\|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big] \\
&\quad + 3\|(E_{x,h}^{0})^2 + (E_{y,h}^{0})^2\|^2_{\varepsilon_0\chi^{(3)}}\Bigg) \times \exp\left(4\Delta t \sum_{n=1}^{N}(1-4\Delta t)^{-1}\right).
\end{aligned}
$$

For $\Delta t \leq \frac{1}{8}$, this leads finally to

$$\begin{aligned}
&\|E_{x,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} \\
&+ 3\|(E_{x,h}^{N+1})^2 + (E_{y,h}^{N+1})^2\|^2_{\varepsilon_0\chi^{(3)}} \\
&\leq \Big(\Delta t \sum_{n=1}^{N} \Big[J_{x,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \|J_{y,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} \Big] \\
&\quad + \frac{5}{4}\Big[\|E_{x,h}^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{y,h}^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big] \\
&\quad + 3\|(E_{x,h}^0)^2 + (E_{y,h}^0)^2\|^2_{\varepsilon_0\chi^{(3)}}\Big) \times \exp(8T+1).
\end{aligned}$$

Since the term $\Delta t \sum_{n=1}^{N} \Big[\|J_{x,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}} + \|J_{y,h}^{n+\frac{1}{2}}\|^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}}\Big]$ can be regarded as an approximation to $\|\mathbf{J}\|^2_{\mathbf{L}^2(0,T,\mathbf{L}^2_{[\varepsilon_0(1+\chi^{(1)})]^{-1}}(\Omega))}$, it is bounded. $\square$

4.4 Summary

In this chapter, a TDdG has been developed for nonlinear Maxwell's equations in Optics and Photonics. The new capabilities of the proposed method permit that linear and nonlinear effects of the electric polarization in 2D are modeled in an efficient manner that is unconditionally stable and conserves the energy. The novel approach allows energy stability both at the semi-discrete and fully discrete levels, which were not yet available for the full system of nonlinear Maxwell's equations in 2D. An error estimate is provided for the semi-discrete problem. The approach is almost completely general and could replace the electric field formulation, magnetic field formulation, and A-formulation.

Appendix

5.1 Some Identities and Inequalities

Lemma 5.1 *Let X be a real Hilbert space with inner product $(\cdot,\cdot)$. Then the following relations are valid for all* $\mathbf{u},\mathbf{v}\in X$*:*

1. $2(\mathbf{u}-\mathbf{v},\mathbf{u}) = \|\mathbf{u}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 - \|\mathbf{v}\|^2$,
2. $|(\mathbf{u},\mathbf{v})| \le \dfrac{\alpha}{2}\|\mathbf{u}\|^2 + \dfrac{1}{2\alpha}\|\mathbf{v}\|^2$ *for all* $\alpha > 0$,
3. $|(\mathbf{u},\mathbf{v})| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$,
4. $(\mathbf{u}+\mathbf{v})^2 \le 2\big[(\mathbf{u})^2 + (\mathbf{v})^2\big]$.

Proof: (1) follows from

$$\|\mathbf{v}\|^2 = (\mathbf{v}-\mathbf{u}+\mathbf{u},\mathbf{v}-\mathbf{u}+\mathbf{u}) = \|\mathbf{v}-\mathbf{u}\|^2 + 2(\mathbf{v}-\mathbf{u},\mathbf{u}) + \|\mathbf{u}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2 - 2(\mathbf{u}-\mathbf{v},\mathbf{u}) + \|\mathbf{u}\|^2.$$

(2) Obviously, $0 \le \|\mathbf{u}\pm\mathbf{v}\|^2 \le \|\mathbf{u}\|^2 \pm 2(\mathbf{u},\mathbf{v}) + \|\mathbf{v}\|^2$, hence $2|(\mathbf{u},\mathbf{v})| \le \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Replacing in this inequality $\mathbf{u}$ by $\sqrt{\alpha}\mathbf{u}$ and $\mathbf{v}$ by $\mathbf{v}/\sqrt{\alpha}$, the statement follows.

(3) is the Cauchy-Schwarz inequality.

(4) follows from $(\mathbf{u}+\mathbf{v})^2 = \|\mathbf{u}\|^2 + 2(\mathbf{u},\mathbf{v}) + \|\mathbf{v}\|^2$ and (2) with $\alpha = 1$. $\square$

Lemma 5.2 *Let (a_n) and $(c_{n-\frac{1}{2}})$ be real sequences. Then it holds, for all $N = 2, 3, 4, \ldots$:*

$$\sum_{n=1}^{N} c_{n-\frac{1}{2}}\left[a_n - a_{n-1}\right] = c_{N-\frac{1}{2}}a_N - c_{\frac{1}{2}}a_0 + \sum_{n=1}^{N-1}\Big[c_{n-\frac{1}{2}} - c_{n+\frac{1}{2}}\Big]\,a_n.$$

Proof: The proof runs by induction. For $N = 2$, it is easily to see that

$$c_{\frac{1}{2}}\left[a_1 - a_0\right] + c_{\frac{3}{2}}\left[a_2 - a_1\right] = c_{\frac{3}{2}}a_2 - c_{\frac{1}{2}}a_0 + \left[c_{\frac{1}{2}} - c_{\frac{3}{2}}\right]a_1.$$

Now we suppose that the formula is true up to some $N \geq 2$. Then we have

$$\begin{aligned}
&\sum_{n=1}^{N+1} c_{n-\frac{1}{2}}\left[a_n - a_{n-1}\right] = \sum_{n=1}^{N} c_{n-\frac{1}{2}}\left[a_n - a_{n-1}\right] + c_{N+\frac{1}{2}}\left[a_{N+1} - a_N\right] \\
&= c_{N-\frac{1}{2}}a_N - c_{\frac{1}{2}}a_0 + \sum_{n=1}^{N-1}\left[c_{n-\frac{1}{2}} - c_{n+\frac{1}{2}}\right]a_n + c_{N+\frac{1}{2}}\left[a_{N+1} - a_N\right] \\
&= c_{N+\frac{1}{2}}a_{N+1} - c_{\frac{1}{2}}a_0 + \sum_{n=1}^{N-1}\left[c_{n-\frac{1}{2}} - c_{n+\frac{1}{2}}\right]a_n + c_{N-\frac{1}{2}}a_N - c_{N+\frac{1}{2}}a_N \\
&= c_{N+\frac{1}{2}}a_{N+1} - c_{\frac{1}{2}}a_0 + \sum_{n=1}^{N}\left[c_{n-\frac{1}{2}} - c_{n+\frac{1}{2}}\right]a_n.
\end{aligned}$$

□

Lemma 5.3 *Let $\delta \geq 0$, $g_0 \geq 0$ and (a_n), (b_n), (c_n) and (γ_n) be sequences of nonnegative numbers such that*

$$a_N + \delta\sum_{n=0}^{N} b_n \leq \delta\sum_{n=0}^{N}\gamma_n a_n + \delta\sum_{n=0}^{N} c_n + g_0 \quad \textit{for all } N = 0, 1, 2, \ldots \tag{5.131}$$

Assume that $\gamma_n\delta < 1$ for all n, and set $\sigma_n := (1 - \gamma_n\delta)^{-1}$. Then it holds, for all $N = 0, 1, 2, \ldots$:

$$a_N + \delta\sum_{n=0}^{N} b_n \leq \left(\delta\sum_{n=0}^{N} c_n + g_0\right)\exp\left(\delta\sum_{n=0}^{N}\sigma_n\gamma_n\right).$$

Proof:See [61, Lemma 5.1]. □

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Publications and Conference Proceedings

This thesis was written from these publications.

1. A. Anees and L. Angermann, "Energy conserving discontinuous Galerkin time domain finite element method for nonlinear model in Optics and Photonics,", (in preparation).

2. A. Anees and L. Angermann, "Energy Stability and Convergence of a Time Domain Finite Element Method for the 3D Nonlinear Maxwell's Equations," IEEE Photonics Journal, (under review, PJ-009942-2019).

3. A. Anees and L. Angermann, "Time Domain Finite Element Method for Maxwell's Equations," IEEE Access, vol. 7, pp. 63852-63867, 2019.doi: 10.1109/ACCESS.2019.2916394.

4. A. Anees and L. Angermann, "time domain finite element methods for Maxwell's equations," Parallel in time integration 7th workshop, pp. 1-32, May, 2nd– 5th Rosocff Marine Station, France.

5. A. Anees and L. Angermann, "Time Domain Finite Element Methods for Maxwell's Equations in Three Dimensions," IEEE and Applied Computational Electromagnetics Society (ACES) Symposium, pp. 1-2, March 24th -29th, 2018, Denver USA.

6. A. Anees and L. Angermann, "Mixed Finite Element Methods for the Maxwell's Equations with Matrix Parameters," IEEE and Applied Computational Electromagnetics Society (ACES) Symposium, pp. 1-2, March 24th -29th, 2018, Denver USA.

7. A. Anees and L. Angermann, "A mixed finite element method approximation for the Maxwell's equations in Electromagnetics," IEEE International Conference on Wireless Information Technology and Systems (ICWITS) and Applied Computational Electromagnetics Society (ACES), pp. 1-2, March 13th -18th 2016.

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2. "Time Domain finite element method for Maxwell's equations," 9th international HPC School 2018, July 8th -13th , Ostrava Czech Republic.

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